Referee Report

on "Non-Hermitian Holographic Flows to Little Rip Cosmologies"

arXiv:2607.14284

Summary of the paper

The manuscript studies the interiors of neutral planar black holes in a holographic model for a non-Hermitian, $\mathcal{PT}$-symmetric deformation of a three-dimensional CFT. The boundary sources of a charged scalar operator are unequal, while the bulk action remains the conventional Einstein–Maxwell–scalar action. With the homogeneous ansatz of equation (3.9), the scalar stress tensor is proportional to $1-\eta^2$. The real solutions with $|\eta|<1$ therefore satisfy the null energy condition (NEC) and are related by a complexified $U(1)$ transformation to a Hermitian model. At nonzero temperature the model also has real solutions for $1<|\eta|<\eta_c$. These violate the NEC and are called the $\mathcal{PT}$-restored phase. The exterior phase diagram and the two real branches are reviewed from earlier work.

The main new work concerns the black-hole interior. The ordinary $\mathcal{PT}$-symmetric solutions and part of the subdominant restored branch approach scalar-supported Kasner regimes. In the dominant restored branch, and in another interval of the subdominant branch, the original radial coordinate instead reaches a surface $z=z_\sigma$ at which the fields have half-integer expansions. The authors introduce $u=\sqrt{z_\sigma-z}$, argue that $u=0$ is a regular surface, and continue the real solution to $u<0$. On this second sheet the coordinate $\tilde{z}$ decreases toward zero, the scalar and f diverge, and g approaches a negative constant.

For a positive quartic coupling, the proposed second-sheet endpoint has

$$f\sim-\frac{32v}{3}(\log\tilde{z})^2,\qquad \psi^2\sim-\frac{8}{\eta^2-1}\log\tilde{z}.$$

In comoving variables this gives an isotropic FLRW metric with $a(\tau)\sim\exp[\exp(\alpha\tau)]$. With a purely quadratic potential the authors instead obtain $a(\tau)\sim\exp(\alpha\tau^2)$. Both geometries have curvature growing without bound only as $\tau\to\infty$, and are interpreted as Little Rip cosmologies rather than finite-proper-time singularities. Section 5 then considers radial spacelike geodesics joining the two boundaries. The claimed singularity-sensitive terms in their lengths scale as $1/\log E$ for $v>0$ and $1/\sqrt{\log E}$ for $v=0$, in contrast with the power-law corrections associated with Kasner interiors.

Section 2 supplies a finite-dimensional motivation for the thermal state. It constructs a biorthogonal Gibbs operator and a doubled state for the standard two-level $\mathcal{PT}$-symmetric Hamiltonian, and argues that the special thermal/TFD state is $\mathcal{PT}$-invariant even when the single-copy spectrum is in the broken regime. This construction is then used to motivate the real finite-temperature saddles and their two-sided interpretation.

Evaluation

The central geometric observation is interesting and, within the literature cited in the manuscript, appears genuinely new. A real, NEC-violating black-brane saddle whose interior passes through an areal-radius bounce and approaches a Little Rip FLRW regime would be a valuable addition to the holographic-interior literature. The dependence of the asymptotic form on the scalar potential and the possibility of isolating this behavior in analytically continued boundary correlators are also potentially significant.

Several important parts of the analysis withstand direct checks. Equations (3.14)–(3.16) follow from the reduced bulk action, and the displayed UV coefficients in equation (3.17) satisfy those equations through the stated orders. For the real ansatz,

$$T_{MN}\ell^M\ell^N = 2(1-\eta^2)(\ell^z\psi')^2$$

for a radial null vector, so the NEC distinction between the two real regimes is correct. The leading balances in equations (5.1)–(5.4) are also consistent for $v > 0$ and $\eta^2 > 1$, as are the leading quadratic-potential balances in equations (B.1)–(B.6) for the mass used in the paper. In particular, the local asymptotic metrics do have the advertised double-exponential and Gaussian-exponential scale factors.

The present version is nevertheless not ready for publication. The proposed boundary diagnostic contains elementary normalization and kinematic errors: equation (5.13) has the wrong UV counterterm sign, equations (5.17) and (B.14) omit a factor of the radial coordinate, and the manuscript repeatedly identifies a near-singularity correction with the full renormalized geodesic length. These issues touch one of the principal claims in the abstract and conclusions. The real continuation through z_σ, which is the other central claim, also needs a coordinate-invariant and numerically controlled justification. In addition, the effective-fluid formulas are internally inconsistent, the numerical precision and convergence evidence are insufficient, and the thermal/TFD construction requires a more precise physical interpretation in the broken spectral regime.

I believe that the qualitative Little Rip result can survive these corrections. The paper could then be suitable for JHEP or PRD, but this requires a substantial revision rather than a collection of small amendments.

Recommendation

I recommend **major revision**. A publishable revision should correct and recompute the complete geodesic observable, establish the real second-sheet extension and its Lorentzian signature invariantly, repair the FLRW stress tensor, provide reproducible convergence tests and broader parameter coverage, and sharply define the non-Hermitian thermal state and the meaning of $\mathcal{PT}$ restoration. The final claims should distinguish a novel subleading signature on an analytically continued geodesic branch from the full physical two-sided correlator.

Major comments

1. **The heavy-operator observable is not correctly derived in the present version.**

   There are several separate problems in Section 5.3, and together they require the calculation to be redone from equations (5.12)–(5.13), not merely rephrased.

   First, the UV subtraction in equation (5.13) has the wrong sign. Near the AdS boundary $f = g = 1 + O(z^2)$, so its doubled bare integral behaves as

   $$2\int_\epsilon \frac{dz}{z} = -2\log\epsilon + \text{finite}.$$

   The renormalized length must therefore contain $+2\log\epsilon$, up to an immaterial finite scheme constant. The printed term $-2\log\epsilon$ doubles the divergence instead of cancelling it.

Second, the boundary-time integrand does not follow from equation (5.12). Combining the conserved energy and normalization conditions gives, up to the sign associated with radial orientation,

$$\frac{dt}{dz} = \frac{Ez}{g}\sqrt{\frac{g}{f(E^2z^2+g)}}.$$

Equation (5.17) omits the factor z, and equation (B.14) repeats the same omission. The printed integrand does not approach the radial-null integrand at fixed z as $E \to \infty$. The coefficient and sign of $t_{\text{bdry}} - t_{\text{sing}}$, and hence equations (5.18) and (B.15), must be rederived after fixing the orientation and contour conventions.

Third, equations (5.15)–(5.16) compute only a contribution from the Little Rip end of the integral. They do not give the complete $\mathcal{L}_{\text{ren}}$. With the correct UV subtraction, the near-boundary region supplies the familiar leading behavior

$$\mathcal{L}_{\text{ren}}(E) = -2\log E + C + \mathcal{L}_{\text{LR}}(E) + \cdots,$$

where C and analytic corrections depend on the rest of the flow, while $\mathcal{L}_{\text{LR}}$ is the singularity-sensitive term. Thus the full length does not vanish as $1/\log E$. The new information is a subleading nonanalytic correction. This is the same conceptual separation made in the cited Kasner analysis of Frenkel et al. The abstract, the discussion after equation (5.16), and the conclusions should all be revised. The full correlator should be displayed as its universal short-distance singularity multiplied by, or corrected by, the Little Rip contribution.

Fourth, the coefficient of that correction is presently ambiguous. Equation (5.13) has an explicit factor of two for the two symmetric radial legs, whereas equation (5.15) has no factor of two. It is not stated whether $\mathcal{L}_{\text{sing}}$ is one leg or the complete singular contribution. The second integral in equation (5.13) also runs from z_σ down to $\tilde{z}_* < z_\sigma$; with the principal positive square root its sign is opposite to a positive proper length. The later use of $\log\tilde{z} < 0$ effectively chooses another branch. The radial orientation, square-root branch, and normalization must be made explicit and used consistently.

Finally, this geodesic is generally reached on an analytically continued branch of the two-sided correlator. In the Kasner literature cited by the authors, the near-singularity saddle is not the dominant saddle on the physical sheet. The revision should state the horizon-pole prescription, the imaginary part of the boundary-time separation, the precise combination of t_L and t_R, and whether the real continued geometry preserves an energy-independent TFD separation. It should also say whether the saddle is physical, subdominant, or available only after analytic continuation. A numerical evaluation of the complete integrals is essential: plots of $t(E)$, $\mathcal{L}_{\text{ren}}(E) + 2\log E$, and the residual after subtracting analytic terms should test the predicted $1/\log E$ and $1/\sqrt{\log E}$ behaviors against a Kasner solution.

2. **The real continuation across $z = z_\sigma$ needs a coordinate-invariant derivation.**

The use of $u = \sqrt{z_\sigma - z}$ is plausible and the half-integer series in equation (4.4) is compatible with a locally smooth field configuration in u. However, equation (4.5) itself has an unresolved sign problem. If

$$f = Au^2 + f_{3/2}u^3 + \cdots, \qquad z = z_\sigma - u^2,$$

direct substitution into equation (3.9) gives

$$\frac{dz^2}{f} = \frac{4u^2}{Au^2 + f_{3/2}u^3 + \cdots}\,du^2.$$

Equation (4.5) instead includes an additional explicit minus sign and introduces f_1 without defining its relation to the coefficient in equation (4.4). Since the numerical solution approaches $f = 0$ from below, $A < 0$, and the expression above already has the required timelike sign. The printed expression can have the wrong signature. All coefficient and sign conventions should be stated, and the metric should be shown explicitly to be nondegenerate and Lorentzian at $u = 0$.

More generally, the manuscript currently moves from the existence of a convenient analytic continuation to the assertion that it is the physically relevant continuation. The revision should write the field equations directly in regular u variables, establish local uniqueness of the real continuation from data at $u = 0$, and explain whether other real global completions are possible. The induced metric, scalar data, and extrinsic curvature should be evaluated on both sides rather than inferred only from the absence of fractional powers in u. At least two independent curvature invariants should be shown finite at the matching surface. A causal diagram or an equivalent precise description of the areal-radius bounce, horizon, second sheet, and future endpoint would make the global construction much clearer.

There is also a related sign issue in the definition of interior proper time. Inside the horizon $f < 0$. Equation (4.2), the definition preceding equation (5.5), and its Appendix-B analogue should therefore use

$$d\tau = \pm \frac{dz}{z\sqrt{-f}}$$

if τ is real and the radial term is $-d\tau^2$. As written with $\sqrt{f}$, the coordinate is imaginary. This should be corrected throughout.

3. **The effective FLRW stress tensors are algebraically inconsistent with the displayed metrics.**

   For the quartic case, equations (5.5) and (5.7) give asymptotically

$$a(\tau) = \exp[\exp(\alpha\tau)], \qquad H = \alpha\exp(\alpha\tau) = \alpha\log a, \qquad \dot{H} = \alpha H.$$

   With the standard flat-FLRW convention $8\pi G = 1$,

$$\epsilon = 3H^2 = 3\alpha^2(\log a)^2, \qquad p = -\epsilon - 2\dot{H} = -\epsilon - 2\alpha\sqrt{\frac{\epsilon}{3}}.$$

   Equation (5.9) agrees with the second formula, but equation (5.10) is too small by a factor of four. The two printed expressions also fail the continuity equation $\dot{\epsilon} + 3H(\epsilon + p) = 0$.

   Appendix B contains two analogous errors. Equation (B.7) gives

$$a(\tau) = \exp(\alpha\tau^2), \qquad H = 2\alpha\tau, \qquad \dot{H} = 2\alpha.$$

   The corresponding formulas are

$$\epsilon = 12\alpha\log a, \qquad p = -\epsilon - 4\alpha,$$

   not equations (B.9)–(B.10). These discrepancies cannot be repaired by one overall convention for Newton's constant because the printed pairs violate stress-tensor conservation. If a different definition of the effective stress tensor is intended, it must be stated and derived consistently.

   The Little Rip classification itself remains plausible after these corrections: in both cases H and ϵ grow without bound, $w \to -1$ from below, and the endpoint lies at infinite proper time. The manuscript should nevertheless treat equations (5.5)–(5.10) and (B.5)–(B.10) as asymptotic relations, since logarithmic subleading terms have been omitted. Section 5 should also say $v > 0$, not merely $v \neq 0$: the stated real value $\alpha = 4\sqrt{2v/3}$ and the required sign of f assume positive v. The case $v < 0$ is not covered by the argument.

4. **The numerical support is not yet adequate for the precision and phase-wide claims.**

   Appendix C says that the calculation uses "15×MachinePrecision", but this is not an unambiguous specification of a Mathematica computation. The paper should give the literal values or code for `WorkingPrecision`, `AccuracyGoal`, `PrecisionGoal`, precision tracking, and the `NDSolve` method. At ordinary machine precision, $1 + 10^{-20}$ is indistinguishable from 1, a Newton update threshold of 10^{-50} is meaningless, and evolution to $\tilde{z} = 10^{-150}$ cannot support quantitative conclusions. Even if the intended working precision is roughly 240 decimal digits, a change in Newton iterates is not an equation residual.

   The revision should provide spectral convergence under changes of the 100-point Chebyshev grid; maximum residuals of equations (3.14)–(3.16); sensitivity to working precision and solver goals; stability under changes of the 10^{-20} horizon and matching offsets; and the fit window, truncation order, and uncertainty for z_σ, g_0, $f_{3/2}$, and ψ_0. Relative-error or residual panels should accompany Figures 6 and 7, since visually superposed curves on a logarithmic axis do not quantify agreement.

   There are also concrete transcription errors in the numerical input. Equation (C.1) contains an undefined x where the equations of motion require η, and the denominator of equation (C.3) contains ψ_0^4 where horizon regularity requires ψ_h^4. The authors should correct these expressions and explicitly confirm that the implementation used the corrected formulas.

   The parameter coverage is narrow. Figures 3, 4, and 6 concern $v = 1$, $T/M = 0.5$, and Figures 4 and 6 use essentially one value, $\eta = 1.45$; Figure 7 gives one $v = 0$ example. Yet the text makes claims about the entire dominant branch and a finite interval of the subdominant branch. This is particularly delicate because the quoted values $\eta_s \simeq 1.47$ and $\eta_c \simeq 1.49$ differ by only about 0.02. The revision should quote uncertainties, define an operational test of Kasner stabilization, and scan several temperatures, positive quartic couplings, and values on both branches. A table of $\eta_s(T/M, v)$, $\eta_c(T/M, v)$, z_σ, and g_0 would be useful. Otherwise the global statements should be restricted to the cases actually demonstrated.

5. **A formal leading asymptotic solution is not yet a universality or attractor result.**

   Equations (5.1)–(5.4) do solve the leading dominant balances for $v > 0$, and equations (B.1)–(B.6) do so for $v = 0$. This is an important positive result. However, the statement that g_0 is the only free coefficient "at this order" does not show that generic data on the claimed branches approach these asymptotics. A linearized perturbation analysis around each Little Rip solution should count the independent modes, identify growing and decaying perturbations, and establish the dimension of its basin of attraction. Such an analysis would also clarify whether the observed endpoint is forced by regular horizon data or is a special numerical trajectory.

   The $v \to 0^+$ limit is nonuniform: the quartic result gives a double exponential and a $1/\log E$ correction, whereas the exactly quadratic theory gives a Gaussian exponential and a $1/\sqrt{\log E}$ correction. The crossover scale and order of limits should be discussed. The conclusions currently say only that the length has an inverse-logarithmic behavior, which is not true for the quadratic model.

   Finally, g_0 can be removed locally from the FLRW line element by rescaling the interior t direction, but it is not globally inessential once the boundary normalization $g(0) = 1$ has fixed the time coordinate. Indeed, g_0 appears in the proposed boundary signature. The manuscript should distinguish a local coordinate normalization from the physical UV–IR matching datum.

6. **The non-Hermitian thermal state and the meaning of $\mathcal{PT}$ restoration require a more precise foundation.**

   There is first a concrete error in the toy model. For equation (2.1), the Hamiltonian eigenvalues are $E_\pm = \pm\sqrt{g^2 - \Gamma^2}$. The commutator superoperator appearing in equation (2.8) therefore has eigenvalues

$$0, \quad 0, \quad \pm 2\sqrt{g^2 - \Gamma^2},$$

not just $\pm\sqrt{g^2 - \Gamma^2}$. The instability conclusion survives, but its rate is wrong by a factor of two. More importantly, in the broken regime let $\kappa = \sqrt{\Gamma^2 - g^2}$. Equation (2.5) then has

$$\mathcal{Z} = 2\cos(\beta\kappa).$$

This quantity has zeros and changes sign. The formal biorthogonal Gibbs operator is consequently undefined at a discrete set of temperatures and is not a positive density matrix in the ordinary sense. A positive-definite metric operator also cannot turn a Hamiltonian with complex eigenvalues into a Hermitian one. The claim after equation (2.7) that the result holds generally for real β is therefore too broad.

The authors should specify whether their object is a physical density matrix, an indefinite-metric ensemble, or a non-Hermitian transition matrix; define the inner product and allowed parameter range; and derive the trace, partial trace, TFD dual, and expectation-value rules in one consistent formalism. The usual commutator evolution in equation (2.8) is not the generic evolution law for a density matrix under an effective non-Hermitian Hamiltonian, so that choice also needs justification. The identical left-copy Hamiltonian with reversed orientation should be compared with the antiunitarily transformed or dual generator that time reversal normally produces. The generalized non-Hermitian TFD and transition-matrix formalism of Guo and Liu, arXiv:2406.06961, is directly relevant and should be discussed.

These distinctions should then be carried into Section 3.1. The deformation may be $\mathcal{PT}$-symmetric, the single-copy spectrum may be in a broken regime, the special doubled state may be $\mathcal{PT}$-invariant, and a real gravitational saddle may exist; these are not the same statement. Likewise, "thermodynamically dominant" means dominance among the Euclidean saddles being compared, not necessarily a stable equilibrium phase. The manuscript itself notes that the restored solutions in the actual charged model have an unstable quasinormal mode. That instability should be visible in the interpretation of the proposed correlator and in the strength of the claims.

7. **The comparison with Kasner singularities and the statement of completeness need qualification.**

   Section 5.1 says that the Ricci scalar of a Kasner singularity behaves as $1/\tau^2$. This is not a universal diagnostic. For the scalar-supported metric in equation (4.1), its leading coefficient is proportional to $(1-\eta^2)p_\psi^2$, and it vanishes in the vacuum Kasner case. Schwarzschild–AdS has a divergent Kretschmann scalar at its Kasner endpoint while its Ricci scalar remains fixed by the cosmological constant. The manuscript should either qualify the comparison as applying to the nontrivial scalar-supported solutions or use a curvature invariant such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ that captures the vacuum case.

   Infinite comoving proper time also does not by itself prove geodesic completeness. For the asymptotic FLRW geometries the desired statement appears to be true toward the future endpoint: timelike proper time grows as τ, while the affine parameter of a radial null geodesic is proportional to $\int^\infty a(\tau)\, d\tau$, which diverges for both scale factors. This short argument, extended to general conserved spatial momentum, should be included. The conclusion should say "future geodesically complete toward the Little Rip endpoint" rather than making an unqualified global claim.

   The relation to BKL should also be calibrated carefully. BKL describes generic, usually inhomogeneous evolution toward a finite-proper-time spacelike singularity. The manuscript studies a homogeneous and isotropic ansatz whose new endpoint lies at infinite proper time. The result is best described as avoiding the usual BKL endpoint through an NEC-violating Little Rip regime, not as establishing a new generic class within BKL dynamics. No stability analysis against anisotropic or inhomogeneous perturbations is given.

8. **The novelty is plausible, but its precise scope and literature context should be sharpened.**

   The model, the exterior phase diagram, and the existence of the two real finite-temperature branches come from earlier work. The new contribution is specifically the real interior continuation, the Little Rip asymptotics, and their proposed geodesic correction. The introduction and conclusions should identify these components explicitly. The simultaneous supergravity-flow and wormhole constructions of Begines et al. are already cited, but a more direct comparison would help distinguish their real-metric analytic continuations from the present black-hole interior.

   The Little Rip discussion is presently based almost entirely on the original reference. The authors should compare their effective phantom sector with the scalar criteria and models in *Models for Little Rip Dark Energy*, arXiv:1108.0067, and with the phantom-scalar construction in *Classical and quantum cosmology of the little rip abrupt event*, arXiv:1604.08365. Prior AdS black holes supported by phantom scalars, for example arXiv:1707.04429, are also useful context for explaining what the non-Hermitian boundary origin adds. This comparison would strengthen, rather than weaken, the genuinely interesting holographic claim.

   At the same time, “Little Rip/CFT correspondence” is premature for one class of unstable saddles and one subleading, analytically continued geodesic correction. A more accurate conclusion would present this work as a first holographic realization and a candidate diagnostic, conditional on the corrected continuation and correlator analysis.

Minor comments

1. Equations (B.11)–(B.13) mix the original z coordinate with the second-sheet $\tilde{z}$ coordinate. In particular, equation (B.12) takes $z_* \to \infty$, whereas equation (B.13) and the turning-point condition require $\tilde{z}_* \to 0$. This appears to be a remnant of the Kasner-coordinate calculation and should be corrected.

2. Figures 6(a) and 7(a) show a positive quantity labelled g, although equations (5.1) and (B.1), as well as the continued branch in Figure 4(a), require $g \to -g_0 < 0$. The plotted quantity appears to be $-g$ and should be labelled accordingly.

3. State explicitly that $g_0 > 0$ and define the rescaled spatial coordinate inherited from the original t direction in equations (5.7) and (B.7).

4. Equation (A.3) is followed by a definition of $S_{\text{ren,E}}$, although that quantity does not appear in the displayed formula. Please state the relation among the Euclidean action, F, and the free-energy density denoted by lowercase f in Figure 2. The Hawking-temperature formula used to fix T/M should also be given.

5. An NEC inequality is not defined in the usual real-ordering sense for the complex metric and stress tensor of the $\mathcal{PT}$-broken branch. The wording in Sections 3.1 and 6 should distinguish analytic continuation of the NEC-violating real branch from a literal inequality in the complex saddle.

6. The values denoted by η_s and η_c depend on T/M, v, and the chosen quantization. This dependence should be visible in the notation or stated whenever the numerical values are quoted.

7. Equations (5.5) and (B.5) should use an asymptotic equality, and the ellipses should indicate which corrections are small in the $\tau \to \infty$ limit.

8. The captions of Figures 6 and 7 should give the fitted values of the asymptotic coefficients and the fitting interval. Figure 3 should show the two Kasner-relation residuals in addition to the extracted exponents.

9. Please correct the remaining typographical problems, including “Einsten–Maxwell–scalar”, “can be casted”, the duplicated “in 3d” phrasing where applicable, and inconsistent singular/plural agreement in the figure captions.