Referee Report

Dynamical Entropy Is a Noether Charge

Recommendation: Major revision

Summary and overall assessment

The manuscript proposes a dynamical entropy density for a generic expanding null hypersurface in Einstein gravity. The construction starts from the Einstein–Hilbert action supplemented by a null boundary term, imposes Dirichlet data for the induced spatial metric and null generator, and selects a future-directed vector field ξ through the four kinematic conditions in equations (14a)–(14d). Writing $\xi^\mu = \kappa\mathscr{B}l^\mu$ on the null surface, the authors obtain the candidate entropy density

$$\mathcal{S} = \frac{\sqrt{q}}{4G}\big(1 - \mathscr{B}\theta_l\big)$$

in equation (25). Equations (26)–(29) then combine the proposed dynamical zeroth law with the Raychaudhuri equation to give the local balance identity

$$l \cdot \nabla\mathcal{S} = \frac{\sqrt{q}}{4G}\mathscr{B}\left(N_l^2 + \frac{\theta_l^2}{D-2} + R_{ll}\right).$$

If $\mathscr{B} \geq 0$ and the null energy condition holds, the right-hand side is nonnegative without imposing a future stationary endpoint.

The aim is important. A genuinely nonlinear, nonteleological entropy balance law on a general null boundary would be valuable, and the attempt to tie the choice of entropy to a variational principle rather than simply postulate it is potentially useful. The chain from equation (26) through equation (29) is also algebraically sound once all of its assumptions are granted.

I do not, however, think that the main claims are established in the present version. There is a direct density mismatch between equations (23) and (24); the boundary term responsible for the correction is an endpoint term in the affine gauge used in the paper; the generator equations are not compatible with a generic angle-dependent expansion and retain integration data that change the entropy; and the vector field does not preserve the strict Dirichlet data under which it is called a symmetry. Furthermore, the asserted exact identification with the Hollands–Wald–Zhang (HWZ) entropy is incompatible with the evolution equation obeyed by $\mathscr{B}$ away from the perturbative regime. These are central issues rather than matters of presentation.

Major comments

1. **Equations (23) and (24) are not mutually consistent as written.**
   Equation (23) places an overall factor $\sqrt{q}$ in the cross-section measure and then defines

   $$\mathcal{W}^\mu = -n^\mu\sqrt{q}\,\theta_l.$$

   Using equation (14c), $\nabla^{[\mu}\xi^{\nu]} = \kappa\epsilon^{\mu\nu}$, together with $\xi^\mu = \kappa\mathscr{B}l^\mu$ and $\epsilon_{\mu\nu}l^\mu n^\nu = -1$, the boundary contribution in equation (23) is proportional to

   $$-\sqrt{q}\,\kappa\mathscr{B}\big(\sqrt{q}\,\theta_l\big) = -q\,\kappa\mathscr{B}\theta_l,$$

whereas equation (24) contains $-\sqrt{q}\,\kappa\mathscr{B}\theta_l$. Equation (24) follows if $\mathcal{W}^\mu = -n^\mu\theta_l$, with no second factor of $\sqrt{q}$, or if a different density convention is used and the measure in equation (23) is changed accordingly. The displayed equations currently mix these conventions.

This point directly affects the advertised entropy formula. The authors should derive the Noether potential from the total action with all form and density weights displayed, state whether $\mathbf{J}^\mu$ and $\mathrm{Q}^{\mu\nu}$ are tensors or tensor densities, and reproduce equation (24) without an implicit cancellation. The derivation should also fix the normalization of the antisymmetrization brackets and the orientation of the binormal. Until this is done, equation (25) does not follow from the displayed Noether charge.

2. **The null boundary term and the claimed removal of charge ambiguities require a complete corner analysis.**

   The paper adopts an affine generator in equation (8). In precisely this gauge,

$$\sqrt{q}\,\theta_l = l\cdot\nabla\sqrt{q},$$

   so the null term in equation (11), integrated along an affine parameter, is an endpoint or corner contribution. The footnote below equation (11) notes the total-derivative property but argues that the term must nevertheless be retained because $\kappa_l + \theta_l$ is the appropriate quantity before the affine gauge is imposed. That observation does not by itself choose how the total derivative is apportioned between the null boundary and its initial and final joints. Standard formulations using a κ_l null-boundary term with explicit joints and formulations retaining $\kappa_l + \theta_l$ can be related by just such a redistribution.

   This matters because the θ_l term is exactly what shifts the Noether potential and produces the correction in equations (24)–(25). Dirichlet data can make a variational principle well posed without eliminating finite intrinsic boundary counterterms or endpoint improvements. The statement below equation (23) that the W and Y ambiguities are completely fixed is therefore not demonstrated by equations (11)–(13). Indeed, the cited null Brown–York analysis treats the addition of the expansion as a total derivative that changes the corner term without changing the Dirichlet variational principle.

   The authors should include all joints, vary the action before choosing the affine gauge, and show explicitly that the entropy on an intermediate cut is invariant under equivalent boundary/corner representatives. They should also state whether all allowed variations preserve $\delta\kappa_l = 0$; imposing $\kappa_l = 0$ before variation is otherwise insufficient. If the answer depends on a selected polarization or counterterm scheme, that choice must be identified as physical input and compared with the Dirichlet and York charges in the existing literature.

3. **The asserted generator does not exist on a generic null surface.**

   The prescribed gradient in equation (20) has nontrivial spatial integrability conditions. Let D_A denote the derivative intrinsic to a cut. From the hypersurface-orthogonal parameterization (9), together with equations (4), (6), and the affine condition (8), one obtains

$$\eta_A - \omega_A = D_A\Psi, \qquad l\cdot\nabla\Psi = 0.$$

   The transverse projection of equation (20) therefore implies

$$D_A \ln(\kappa\mathscr{B}) = D_A\Psi.$$

   Locally on a connected cut, in an adapted affine coordinate λ, the solution must have

$$\kappa\mathscr{B} = e^{\Psi}C(\lambda).$$

The longitudinal projection of equation (20) gives $l \cdot \nabla(\kappa \mathscr{B}) = \kappa$, while equation (21) gives $l \cdot \nabla \ln \kappa = \theta_l$. Hence

$$\kappa = e^{\Psi} C'(\lambda), \qquad \theta_l = \partial_\lambda \ln C'(\lambda).$$

The last expression is uniform over each connected cut. A generic distorted null congruence has angle-dependent expansion; null geodesics emitted normally from a nonspherical convex surface in Minkowski spacetime already provide elementary examples. Such a congruence cannot obey the displayed relations.

The manuscript asserts that equations (14a)–(14d) define ξ on an arbitrary generic null surface, but it does not examine this integrability condition. The authors should either identify a missing identity that invalidates the argument above or sharply restrict the class of admitted null hypersurfaces and foliations. This question is decisive for the genericity claims in the abstract and Discussion.

4. **Even when a solution exists, the generator and entropy are not unique.**

Contracting equation (20) with l^μ and using equation (21) gives

$$l \cdot \nabla \mathscr{B} = 1 - \mathscr{B}\theta_l, \qquad l \cdot \nabla \ln \kappa = \theta_l.$$

These are evolution equations, not initial conditions. They leave data on a reference cut; equivalently, the solution in the preceding comment permits $C(\lambda) \to C(\lambda) + C_0$. This shift leaves κ, equation (21), and all derivative constraints unchanged but changes $\kappa\mathscr{B}$, ξ, and equation (25). It is not the global rescaling discussed below equation (25). The arbitrary longitudinal function f and residual choices of foliation and null frame also require treatment.

A simple exact check is an outgoing light cone in four-dimensional Minkowski spacetime. With affine radius r, $\theta_l = 2/r$, $N_l^2 = 0$, and $R_{ll} = 0$. Equations (20)–(21) give

$$\kappa = \kappa_0 r^2, \qquad \mathscr{B} = \frac{r}{3} + \frac{C}{r^2}.$$

Equation (25) depends on C. Even the regular choice $C = 0$ assigns one third of the Bekenstein–Hawking area density to an ordinary vacuum light cone and gives positive entropy production from its expansion. Other allowed values change both the entropy and its derivative. This example is especially relevant because existing work has used the difference between Dirichlet and York entropies on a Minkowski light cone to expose precisely this charge ambiguity.

The authors should specify reference-cut data, prove uniqueness on the stated domain, and show how $\mathscr{B}$, κ, and $\mathcal{S}$ transform under $l \to e^{\sigma} l$, changes of foliation, and residual reparameterizations. If a preferred condition $\mathscr{B} = 0$ is needed, its geometric meaning must be given on a null surface with no bifurcation cut. Explicit light-cone and Vaidya calculations would be particularly useful.

5. **The relation to HWZ is only perturbative with the definitions given here.**

The HWZ entropy in general relativity is

$$S_{\mathrm{HWZ}}[C] = \frac{A[C]}{4G} - \frac{1}{4G} \int_C V \theta_l \sqrt{q}\, d^{D-2}x,$$

where V is an affine parameter vanishing at the bifurcation surface of the stationary background. In the present manuscript l is affine by equation (8), but $\mathscr{B}$ is not its affine parameter: equations (20)–(21) give $l \cdot \nabla \mathscr{B} = 1 - \mathscr{B}\theta_l$, rather than $l \cdot \nabla V = 1$. Thus $\mathscr{B} = V$ is compatible with the proposed zeroth law only when $\theta_l = 0$ (apart from special points).

For a perturbation with $\theta_l = O(\varepsilon)$ one may have $\mathscr{B} = V + O(\varepsilon)$, so $\mathscr{B}\theta_l = V\theta_l + O(\varepsilon^2)$. This explains agreement at the order for which HWZ was derived, but not the statement after equation (25) that the expressions precisely coincide nonlinearly. The difference may be the paper's most interesting potential contribution: equations (20)–(21) could select a particular nonlinear completion of HWZ. The authors should present the precise dictionary, expand through the first order at which the two expressions differ, and explain why their completion is preferred over other improvements.

This comparison must engage directly with the improved Noether-charge constructions of Rignon-Bret and Visser–Yan. Rignon-Bret in particular already analyzes expansion-corrected Noether entropies, Dirichlet and York polarizations, null-surface balance laws, and the Minkowski light cone. A parenthetical citation does not establish what is new here.

6. **The proposed vector does not preserve the stated Dirichlet boundary data.**

   Equation (12) restricts variations by $\delta q_{\mu\nu} = 0$ and $\delta l^\mu = 0$ on $\mathcal{N}$. Let $F = \kappa\mathscr{B}$, so that $\xi^\mu = F l^\mu$ there. Equation (20) gives $l \cdot \nabla F = \kappa$. Consequently,

   $$\mathcal{L}_\xi q_{AB} = 2F\Theta^l_{AB}, \qquad \mathcal{L}_\xi l^\mu = [Fl, l]^\mu = -\kappa l^\mu.$$

   Both are generically nonzero, whereas equation (12) fixes both pieces of boundary data. Thus the diffeomorphism generated by ξ is not a symmetry of the strict Dirichlet phase space except in special cases. A diffeomorphism of a covariant bulk Lagrangian always has a Noether potential, but that fact alone does not make its cut integral a Hamiltonian boundary charge or a thermodynamic entropy.

   There is a related field-dependence issue: ξ and κ are constructed from the dynamical boundary geometry. Charge variations for field-dependent generators require care and can contain terms absent for a fixed vector. The assertion after equation (25) that the result is the integrable part of a covariant-phase-space charge is deferred to a work in preparation, even though it is essential to the interpretation in this paper. The authors should show here which boundary structure is preserved, whether a compensating or anomalous transformation is intended, and how field-dependent-generator terms enter the charge variation. A work in preparation cannot support this central step.

7. **The dynamical zeroth law is not independently justified.**

   Condition (14d) is imposed rather than derived. In combination with equation (20), it is precisely the condition that changes equation (26) into equation (27), after which Raychaudhuri makes the sign manifest. This yields a correct conditional identity if all preceding equations hold, but it risks making the second-law argument circular: the generator is selected by the condition needed to cancel the sign-indefinite term, and the cancellation is then offered as evidence that the condition is thermodynamic.

   The authors should explain why $\nabla \cdot (\xi/\kappa) = 0$ is a zeroth law rather than a convenient volume-preserving gauge, and whether it follows from a first law, equilibrium limit, regularity principle, or boundary symmetry algebra. The usual zeroth law concerns uniformity of surface gravity across a stationary horizon, whereas equation (21) controls evolution along a generator. The relation between these statements should be demonstrated. Since $\sqrt{q}$ is a cross-section density, $\kappa/\sqrt{q}$ in equation (21) is not an ordinary scalar under arbitrary transverse coordinate changes; a coordinate-free formulation using the area form is needed.

   On a generic null surface there is also no Killing normalization, Euclidean periodicity, or specified thermal state that identifies $\kappa/(2\pi)$ as a physical local temperature. Dividing a chosen Noether-charge density by this quantity is not by itself a thermodynamic derivation. A first-law or integrability calculation in the present paper would substantially strengthen the interpretation.

8. **The positivity assumptions and the meaning of the second law need sharpening.**

   Future orientation of $\xi^\mu = \kappa\mathscr{B}l^\mu$ implies $\kappa\mathscr{B} \geq 0$, not $\mathscr{B} \geq 0$ by itself. Equations (14d)–(16) and (21) divide by κ and take its logarithm, but the manuscript never states that κ is positive and nowhere vanishing. These assumptions must be explicit, and the behavior near a zero or sign change of κ must be addressed. Equation (21) is also obtained by steps that divide by $\mathscr{B}$; a cut on which $\mathscr{B} = 0$ requires a continuity argument. Nor is the entropy density in equation (25) necessarily positive: $1 - \mathscr{B}\theta_l$ can be negative.

   Equation (29) proves nondecrease, not strict increase. Equality occurs, for example, when $\mathscr{B} = 0$, or when $N_l = \theta_l = R_{ll} = 0$. More fundamentally, monotonicity alone does not establish that a charge on every arbitrary null surface is thermodynamic entropy. The flat-light-cone example produces entropy in vacuum purely from congruence expansion. The authors should discuss positivity, equality cases, additivity, reference-cut dependence, and the physical observer or coarse graining associated with the proposed entropy.

9. **The novelty and domain of validity must be stated much more precisely.**

   Several ingredients overlap closely with cited work: HWZ and Visser–Yan use an improved Noether charge to define dynamical entropy; Rignon-Bret studies Noether-current entropy balances and multiple polarizations on general null hypersurfaces; Chandrasekaran et al. derive null Brown–York charges from a Dirichlet variational principle; and work by Adami et al., including two present authors, develops thermodynamics and charges on generic null surfaces. The authors' recent *A New Derivation of Classical Gravitational Second Law of Thermodynamics* is also very close in stated scope. The present manuscript needs a term-by-term comparison of the generator, boundary polarization, charge density, and flux law in these constructions. The unpublished citation cannot carry the central novelty claim.

   The current calculation is restricted to Einstein gravity with cosmological constant, affine l, the boundary data (12), the imposed condition (14d), and matter satisfying the null energy condition. The matter action and boundary conditions are not written even though R_{ll} is interpreted through the matter equations. Higher-curvature theories are explicitly left for future work. Claims about generic gravitational systems, arbitrary codimension-one surfaces, cosmology in general, and Brown–York quasilocal thermodynamics therefore go beyond what has been shown. If the true advance is a candidate nonlinear Einstein-gravity completion of HWZ, stating and validating that narrower claim would make the paper clearer and stronger.

Minor comments

1. The sentence preceding equation (26) states $l \cdot \nabla(\kappa\mathscr{B}) = 1$. Contracting equation (20) with l^μ gives $l \cdot \nabla(\kappa\mathscr{B}) = \kappa$. Equation (26) appears to use the correct relation, but the derivation should be checked and the sentence corrected.

2. Equation (14a) does not by itself guarantee tangency to $\mathcal{N}$, since the transverse normal n^μ is also null. Tangency, or $\xi^\mu\partial_\mu\Phi = 0$, should be imposed before selecting the $\mathscr{A} = 0$ branch in equation (18).

3. The phrase *satisfies the second law strictly* in the abstract should be replaced by *is pointwise nondecreasing under the stated assumptions*, unless a strict-positivity theorem is supplied.

4. The phrase *thermodynamically closed system* is confusing when equation (29) explicitly contains matter and gravitational flux through the boundary. Please distinguish fixed variational data, vanishing symplectic flux, and thermodynamic isolation.

5. Below equation (29), the logical implication is stated backwards. In Einstein gravity the matter NEC implies $R_{ll} \geq 0$ through the field equations, not conversely without further assumptions about those equations.

6. The manuscript should specify whether cross-sections are compact, whether caustics and generator endpoints are excluded, and over what interval the foliation and the sign of $\mathscr{B}$ remain regular.

7. Equation (13) is introduced with *one may verify*, and equations (19)–(20) and (23)–(24) are likewise asserted with little intermediate work. These are the central steps and warrant a detailed appendix.

8. In equation (22), please distinguish exterior divergence of a differential-form Noether current from a coordinate partial derivative of a tensor density. This would also help resolve the density problem in equation (23) and the treatment of a lower-dimensional boundary Lagrangian within a bulk current.

9. The rotation one-form ω_μ in equation (4) should not be called the twist of the l congruence. The latter vanishes under hypersurface orthogonality; the normal-bundle connection is a different object.

10. The Discussion refers to an arbitrary codimension-one surface, whereas all defining equations use a null hypersurface. It should consistently say *null codimension-one surface.* The text should likewise distinguish event horizons, generic null boundaries, and spacelike dynamical horizons.

11. The manuscript has an unmatched grouping brace beginning in the introductory paragraph on HWZ, which produces an end-of-document grouping warning. The preamble also loads several packages more than once. These source issues should be cleaned up.

Recommendation

I recommend **major revision**. The proposed local balance identity is interesting, and the manuscript may contain the seed of a useful nonlinear Einstein-gravity completion of the HWZ construction. Publication in its present form would be premature because the displayed Noether-charge derivation is internally inconsistent at equation (23), the boundary/corner ambiguity is unresolved, the entropy depends on unfixed generator data, and the exact relation to prior dynamical entropies is not established. A revision that resolves these points, includes explicit light-cone and Vaidya tests, and narrows the claims to the demonstrated domain could become a substantial contribution.