Referee Report

"The variational approach to infrared divergencies:
Applications to black hole action integrals and holographic Wilson loops"

`arXiv:2607.14330`

Summary and overall assessment

The manuscript advocates computing renormalized on-shell actions by first renormalizing their variations. On a finite-dimensional family of asymptotic solutions, the boundary symplectic potential is treated as a one-form in the parameters. The central proposed decomposition is equation (14),

$$\Theta = q_n\,\delta\mu_n + \delta(\Theta_0 + \Theta_\infty)\,,$$

where the first term is identified as the finite, non-exact charge–potential part and the remaining terms are exact. Equation (26) is then used to argue that every term with explicit cutoff dependence is exact in field space. The authors subtract the exact terms, impose regularity so that the charges become functions of the potentials, and integrate equation (17) to obtain the action. They illustrate the prescription with Schwarzschild, Schwarzschild–AdS, Taub–NUT/Bolt–AdS, JT, Kerr–Newman, Kerr–Newman–AdS, and STU black holes, followed by rectangular and circular holographic Wilson loops.

The organizing viewpoint is useful. In particular, equation (26) gives a compact explanation, under suitable hypotheses, of why the explicitly cutoff-dependent part of the pulled-back boundary one-form is field-space exact. The examples also make the thermodynamic use of the resulting first-law one-form accessible. A carefully delimited version of this material could be valuable to the gravity and holography communities.

In its present form, however, the manuscript claims considerably more than it proves. The Darboux argument does not establish equation (14) for the independently defined physical charges and potentials, equation (26) does not establish locality of the primitive as a boundary counterterm, and the asserted absence of charge dependence from the divergent primitive is contradicted by two of the manuscript's own examples. In addition, the circular Wilson-loop subtraction has the wrong sign, the rectangular Wilson-loop action is still divergent when the paper sets its boundary term to zero, and several displayed black-hole formulas are mutually inconsistent. These are not merely presentational matters: they bear directly on the proposed prescription and on the checks offered in its support.

Recommendation: major revision. I do not recommend publication in the present form. I would be willing to reconsider a substantially revised manuscript that either proves a precise, appropriately qualified result or recasts the proposal as a reduced on-shell prescription, corrects the worked examples, and makes its novelty relative to the cited variational and holographic-renormalization literature explicit.

Correctness and logical structure

The manuscript contains two logically distinct observations which should be separated more sharply. First, equation (26) controls the explicit radial dependence of the on-shell boundary one-form. Second, equation (14) identifies its finite non-exact part with a prescribed set of physical charge–potential pairs and asserts a source-only, local interpretation of the divergent primitive. The first statement does not by itself imply the second. The examples are useful tests of the thermodynamic one-form, but at present they do not close this logical gap; indeed, Kerr–Newman–AdS and STU expose it.

The standard thermodynamic results reproduced in the one-charge examples are largely convincing once the first-law form is assumed. The Schwarzschild and Schwarzschild–AdS integrations, for example, give the expected free energies and entropies. The more complicated examples are less reproducible because the asymptotic variations leading to equations (65), (76), and (90) are omitted. This omission is consequential given the explicit sign and parameter-dependence problems discussed below.

Novelty and significance

The physical free energies, entropies, and Wilson-loop answers presented in the paper are established results. The manuscript also states that the key exact-variation observation appeared in the authors' cited 2007 work by Andrade, Bañados and Rojas, with an earlier use in the Chern–Simons/holographic-anomaly context, and that variational black-hole actions have been treated in subsequent work. Thus the identifiable research advance is the proposed general algorithm and the short proof around equation (26), together with the synthesis of black-hole and Wilson-loop examples.

That could constitute a useful contribution if the scope and gain over the earlier work were made precise. At present the claimed general theorem is incomplete, while the applications yield no new observable. For the standards of a research journal such as JHEP or PRD, the authors should either formulate and prove a genuinely stronger theorem—including locality, boundary conditions, gauge degeneracies, and the status of the physical canonical variables—or add a nontrivial application in which the method produces a new result and demonstrably simplifies a difficult renormalization problem. Otherwise, the manuscript should be presented more modestly as a systematic and pedagogical reduced-solution method.

Major comments

1. **Equation (26) does not prove the full decomposition in equation (14).**
   Let $\Theta(r) = \int_{K(r)} \theta^r$ denote the boundary one-form pulled back to the chosen solution family. Under the assumptions used in Section 2.3, equation (26) implies schematically

   $$\partial_r \Theta(r) = \delta \left[\int_{K(r)} \mathcal{L} \right].$$

   Integrating in the radial direction shows that the part of $\Theta(r)$ which changes with r is locally exact in parameter space, up to an r-independent one-form. This is a useful result. It does not, however, determine that remaining one-form, nor does it prove that it has the particular form $q_n\delta\mu_n + \delta\Theta_0$ for charges and potentials defined by independent ADM, covariant-phase-space, or Gauss-law constructions.

The appeal to Darboux's theorem on page 8 is insufficient. Darboux's theorem gives some local canonical coordinates for a closed, nondegenerate two-form. It does not show that the already designated physical (q_n, μ_n) are those coordinates. Moreover, a gravitational covariant phase space is presymplectic before gauge directions are quotiented; its rank may change, and global cohomological obstructions need not be absent. The JT example itself contains a trivial parameter direction. The authors should compute the pulled-back two-form and establish, under stated hypotheses,

$$\delta\Theta = \delta q_n \wedge \delta\mu_n,$$

or replace the universal statement by an explicit assumption. The required assumptions on gauge reduction, nondegeneracy, global parameter space, allowed variations, logarithmic branches, and field-dependent coordinates should be stated. The assertion in Section 2.1 that a single asymptotic boundary forces the symplectic current to vanish is also not generally valid; possible horizon, corner, gauge, and topology contributions must be treated rather than inferred away from boundary counting.

2. **Field-space exactness is not the same as a local, covariant boundary counterterm.**

   The primitive obtained by radially integrating equation (26) is naturally a radial integral of the bulk Lagrangian. The manuscript does not show that this primitive can be lifted from the finite-dimensional solution family to an off-shell local functional of the induced boundary fields. Such a lift is essential if B in equation (15) is to define a differentiable field-theory action rather than only a function of integration constants. In the examples, B is repeatedly written directly in terms of M, J, q_I, and related global parameters. Checking the variation along a selected family of solutions is weaker than checking the variational principle for arbitrary admissible perturbations with the prescribed boundary data.

   The authors should either exhibit the local boundary functionals, at least for representative nontrivial examples, or state clearly that the prescription operates only on a reduced solution space. If locality is claimed generally, an additional asymptotic Hamilton–Jacobi or equivalent argument is needed. This point is also essential to the asserted equivalence with holographic renormalization: exactness in parameter space alone does not establish locality, covariance, or the Ward identities of the renormalized generating functional.

3. **The examples contradict the claim that Θ_∞ is independent of charge/state data.**

   Immediately after equation (14), the manuscript asserts that the charges do not occur in $\Theta_\infty(r, \mu)$, acknowledges that no general proof is given, and then states that the property holds in all examples. As the variables are classified in the paper, this is false.

   For Kerr–Newman–AdS, equation (76) contains the divergent primitive

$$\frac{\beta r^3}{2\Xi l^2} - \frac{J^2\beta r}{2M^2\Xi l^2}, \qquad \Xi = 1 + \frac{J^2}{M^2 l^2},$$

   where equations (73)–(74) identify M and J as charges and set $a = J/M$. The divergent term therefore depends explicitly on charge-side parameters. In the STU example, equation (90) includes

$$\frac{2r^2}{3l^2}\,\delta[\beta(q_1 + q_2 + q_3)]\,,$$

and the divergent subtraction in equation (91) contains the q_I. Yet the paragraph following equation (88) expressly counts (m, q_I) as the four charge-side integration constants, distinct from (β, μ_I).

This is a direct internal test of the central claim. The authors must explain whether a and some scalar coefficients q_I are actually part of the boundary source data under the chosen boundary conditions. If so, the earlier counting of charges and potentials, and the form of equation (14), must be revised accordingly. If not, the source-only claim must be withdrawn. It would be especially useful to reconstruct equations (76) and (91) as local functionals of the cutoff fields and explain precisely how normalizable and non-normalizable data enter.

4. **Finite exact terms, scheme dependence, and normalization of the action require a more careful treatment.**

   Subtracting every exact one-form, including the finite $\delta\Theta_0$, is a prescription; exactness alone does not make a term physically irrelevant. Allowed finite local counterterms depending on the sources shift the generating functional and its one-point functions. Canonical transformations can also change the split between $q_n \delta\mu_n$ and an exact term unless the physical charges and ensemble are fixed independently. The manuscript's statement that the finite action is uniquely defined therefore needs qualification.

   The variation also leaves an additive constant on each connected family of regular saddles. Relative constants between disconnected families matter for Hawking–Page transitions and for NUT/Bolt competition. The choices $I[0] = 0$, a smooth flat-space limit, and the Wilson-loop self-energy normalization are additional physical or renormalization data, not consequences of equations (14)–(26). The authors should state a scheme, explain which finite counterterms are allowed by the boundary conditions, and compare with a standard holographic-renormalization scheme. The global AdS/STU case is a natural place to discuss vacuum or Casimir-energy conventions. Relatedly, fixing $\{\beta, \Omega, \mu\}$ is conventionally a grand-canonical, not a canonical, ensemble.

5. **Regularity implies useful integrability relations, but the claimed equivalence is too strong.**

   Once a smooth on-shell action exists on a family parameterized by potentials, equation (18) follows as a Maxwell relation. Conversely, those relations do not ensure absence of a conical defect, an axis or Misner-string singularity, an inadmissible gauge holonomy, or some other global obstruction. Nor is it general that regularity supplies exactly N algebraic relations: hairy solutions can carry moduli, extremal limits can change the rank of the relations, and multiple regular branches can coexist.

   The Taub examples already show that asymptotic fibration and Misner-string conditions must be imposed separately from the local bolt condition. Similarly, treating a conically singular intermediate metric as a solution requires either a distributional curvature contribution or an excision prescription at the tip. The authors should reformulate the logical implication as regularity $\Rightarrow$ integrability under explicit hypotheses, state when the converse may hold, and explain the treatment of the singular intermediate configurations used to keep charges and potentials independent.

6. **The circular Wilson-loop subtraction in equations (113)–(117) has the wrong sign.**

For the regular hemisphere $\rho(z) = \sqrt{\rho_0^2 - z^2}$, the bare action is

$$S_{\text{bare}} = \frac{R^2}{\alpha'} \int_\epsilon^{\rho_0} \frac{\rho_0}{z^2}\, dz = \frac{R^2}{\alpha'} \left(\frac{\rho_0}{\epsilon} - 1 \right).$$

It follows that the counterterm in equation (114) must be

$$B = -\frac{R^2}{\alpha'} \frac{\rho_0}{\epsilon},$$

not the displayed positive expression. With the sign printed in the manuscript, the first line of equation (117) equals $(R^2/\alpha')(2\rho_0/\epsilon - 1)$, so it cannot yield the finite second line.

The same lower-limit orientation reverses the preceding variation. Using the expansion in the manuscript gives

$$\delta S_{\text{bare}} = \frac{R^2}{\alpha'} \left(\frac{\delta\rho_0}{\epsilon} - 3\rho_0\rho_3\, \delta\rho_0 + \cdots \right).$$

Thus the divergent and finite signs in equations (113)–(115) should be reconsidered together. Equation (116) must also retain the sign of Π_z associated with the orientation $z'(\sigma) = -1$ stated in its footnote. The correct subtraction does give $S_{\text{ren}} = -R^2/\alpha'$, but the printed derivation does not.

7. **The rectangular Wilson-loop variation is finite, but the action in equation (104) is not.**

   As $r \to \infty$, the integrand in equation (104) approaches one. The on-shell action therefore contains a linear cutoff divergence $Tr_{\text{cut}}/(\pi\alpha')$. Equation (108) is finite only because that divergence is independent of the separation L. Consequently, the statement $B = 0$ does not render the action finite, and equation (102) cannot be applied directly to the unrenormalized $S(L)$.

   The force formula (110) is nevertheless correct: differentiating removes the L-independent quark self-energy. To obtain equation (112), however, one must also choose an integration constant, conventionally by subtracting the two straight-string masses or by an equivalent boundary counterterm. In backgrounds with another scale this constant can matter for comparing connected and disconnected saddles and for screening transitions. This example should be rewritten as a clear illustration of both the strength and the limitation of the variational method: a finite derivative can determine the force while leaving the divergent and finite L-independent normalization undetermined.

8. **Several black-hole equations are inconsistent with the manuscript's stated conventions.**

   There are at least four definite corrections:

   - With the real Euclidean conventions of equations (62) and (67), $r_+ = M + \sqrt{M^2 + a^2 - Q^2}$ and $a = J/M$. Therefore equation (72) implies

     $$\pi(r_+^2 - a^2) = 2\pi \left(M^2 - \frac{Q^2}{2} + \sqrt{M^4 + J^2 - Q^2 M^2} \right),$$

     whereas equation (71) contains $-J^2$ under the square root. The latter is the Lorentzian sign. If the authors intend to switch to Lorentzian variables at that point, the analytic continuation must be performed explicitly and consistently; otherwise equation (71) must be corrected.

- The Kerr–Newman–AdS action in equation (75) contains $R - 6/l^2$. This is inconsistent with the stated AdS scalar curvature $R = -12/l^2$, with equation (77), and with the Schwarzschild–AdS action (28). In the manuscript's conventions the term must be $R + 6/l^2$.
- Equation (55) drops the curvature radius even though the surrounding equations retain it. The JT equations should read

$$E_\Phi = R + \frac{2}{l^2}, \qquad E^{\mu\nu} = \nabla^\mu \nabla^\nu \Phi - g^{\mu\nu} \nabla^2 \Phi + \frac{1}{l^2} g^{\mu\nu} \Phi.$$

- Equation (29) is not the variation of equation (28) with respect to $g^{\mu\nu}$ as printed. Because equation (28) has the overall factor $-1/(16\pi)$, its bulk term is $-(16\pi)^{-1} \int \sqrt{g}\,(G_{\mu\nu} - 3g_{\mu\nu}/l^2)\delta g^{\mu\nu}$, not the positive term displayed. This vanishes on the cited solutions, but the variational identity should be internally correct.

Given these errors, the omitted algebra leading to equations (65) and (76) should be supplied in an appendix or supplementary file, with coordinate ranges, boundary orientation, normalization, and Euclidean continuation specified. At least one independent symbolic or analytic check would materially improve confidence in the more complicated formulas.

9. **The STU action has a vacuum-sign inconsistency, and it does not test the claimed Chern–Simons generality.**

   At $S = T = U = 1$, equation (83) gives $V = 6/l^2$. With the gravitational terms $-R/2 + V$ printed in equation (82), the vacuum metric equation is $G_{\mu\nu} = -6g_{\mu\nu}/l^2$, whereas the $m = q_I = 0$ limit of equation (84), $f = k + r^2/l^2$, is AdS and obeys $G_{\mu\nu} = +6g_{\mu\nu}/l^2$. The sign convention for the potential or its occurrence in the Euclidean action must be corrected and propagated through the variation.

   Furthermore, the purely electric ansatz (85) has $F^I \wedge F^J = 0$, so the Chern–Simons term in equation (82) is inert. This example therefore cannot substantiate the manuscript's repeated claim that the proposed method has been demonstrated for Chern–Simons interactions. The claim should either be supported by an example with a nonzero Chern–Simons contribution or presented only as an expected extension subject to the quasi-invariance and boundary-term subtleties of such theories.

10. **The Taub–Bolt discussion omits an entire regular branch.**

    Equations (41), (48), and the bolt regularity condition $f'(r_b) = 1/(2n)$ yield

$$6nr_b^2 - l^2 r_b + 2nl^2 - 6n^3 = 0,$$

    and hence

$$r_{b,\pm} = \frac{l^2 \pm \sqrt{l^4 - 48l^2n^2 + 144n^4}}{12n}.$$

    Equation (49) retains only the minus branch without comment, while equations (50)–(51) are presented as the Taub–Bolt result generally. Both admissible branches and their allowed range in n/l should be discussed, or the subsection and the first form of equation (50) should be explicitly labeled as applying only to the lower branch. Since competing branches are precisely where relative action normalizations matter, this omission also reinforces the concern in comment 4.

11. **The novelty claim and reproducibility of the main examples should be strengthened together.**

    The manuscript should compare the precise statement proved by equation (26) with the earlier variational result cited in the Introduction and with the Hamilton–Jacobi logic of holographic renormalization. It should identify what is new beyond a reorganization of known thermodynamic first laws. A nontrivial example in a higher-derivative or genuinely Chern–Simons theory would help, provided the boundary one-form and its local primitive are actually worked out.

    For the present examples, the steps leading from the asymptotic fields to equations (65), (76), and (90) are too compressed. These are the most important checks of the many-charge claim, yet the perturbations and angular integrals are simply omitted. The revision should give enough intermediate expressions to reproduce them and should verify that the same definitions of charges and sources are used in the symplectic form, the counterterm, and the final first law.

Minor comments

1. The standard English plural is "divergences," not "divergencies." This occurs in the title and throughout the text.

2. Equation (62) contains a stray second plus sign in the second squared bracket of the metric.

3. In the STU discussion, saying that a symmetric C_{IJK} has "only" $C_{123} = 1$ nonzero should be replaced by a statement that this is the independent nonzero component and that its permutations are included.

4. The claim near equation (85) that all Euclidean fields and parameters are real is inconsistent with the explicit factor $-i$ in the electric potential. The Euclidean continuation should be stated once and then used uniformly in the Kerr and STU examples.

5. The general force formula should be positioned relative to the foundational Wilson-loop analysis already listed in the bibliography, not only the later reference cited after equation (110).

6. Figure 1 has the placeholder caption "Caption." It should identify the two string branches, cutoff boundary, turning point r_*, and separation L.

7. The circular result is specifically the strong-coupling saddle for the supersymmetric Maldacena–Wilson loop. The scalar coupling and the restriction to the relevant AdS_3 subspace should be stated to avoid conflating it with an arbitrary geometric Wilson loop.

8. The acknowledgments should not be presented as an appendix section. There are also numerous typographical errors (for example, "diffeormorphism," "sympletic," "nostraightforward," "integrtaion," "worlsheet," "intergation," "Eistein," and "renomalization") that require a complete copyedit.

9. The manuscript should distinguish a fixed coordinate cutoff from an invariantly specified cutoff surface. If solution parameters change the asymptotic frame, the variation of the regulator and

of the boundary coordinates can contribute.

10. The compactness assumption on $K(r)$ should be reconciled with planar and hyperbolic STU horizons. One may work with compact quotients or densities, but this choice should be explicit.

Recommendation

The variational perspective has merit, and the exactness statement encoded in equation (26) is potentially a useful organizing principle. The manuscript is nevertheless not ready for publication because its central canonical/local interpretation is not proved, two examples contradict a stated hypothesis, and several displayed checks contain concrete sign or branch errors. I recommend **major revision**. A successful revision should narrow or prove the theorem, correct and make reproducible all worked examples, specify the finite renormalization data, and articulate a research advance beyond the authors' cited earlier variational framework.