Referee Report

Lectures on Open Systems and Cosmology

Enrico Pajer

Recommendation: Major revision.

Summary and overall assessment

This long set of lecture notes develops a bridge between three subjects that are too often presented in different languages: the density-matrix and quantum-channel formulation of open quantum systems, the Schwinger–Keldysh (SK) path integral, and applications to inflation and stochastic dynamics in de Sitter space. Sections 1–3 introduce density matrices, entanglement and decoherence, Kraus maps, complete positivity, the GKSL equation, microscopic Born–Markov–secular approximations, and elementary qubit and oscillator examples. Sections 4–5 formulate the closed-time path, influence functionals, local dynamical maps, Gaussian dissipative systems, jump-operator completions, and Langevin limits. Sections 6–8 then review inflationary correlators, construct the open effective field theory of inflation, and derive the leading stochastic-inflation Langevin and Fokker–Planck descriptions.

The principal value of the manuscript is not a claim that all of these results are new. Its value is the integrated presentation, especially the attempt to connect Kossakowski positivity and CP divisibility to local SK effective actions and then to cosmological observables. This is a timely and potentially very useful resource for high-energy theorists. The manuscript is generally lucid, the selection of examples is good, and many substantial derivations are correct. In particular, the Kraus and GKSL discussion, the population equations for the qubit and oscillator, the jump-operator organization of positivity-improved influence functionals, and the leading stochastic calculations form a coherent pedagogical arc. The agreement between the light-mass result in equations (8.27)–(8.28) and the Fokker–Planck result (8.68), as well as the quartic moment hierarchy and equilibrium analysis in equations (8.74)–(8.82), are useful checks and are presented effectively.

I do not, however, think the notes are ready for publication in their present form. There are several concrete algebraic errors, and there is a convention mismatch in the central SK propagator discussion. In addition, the statistical propagator used for the inflationary power spectrum omits the contribution of the initial density matrix. These issues cut across Sections 4, 5, and 7 and affect the advertised bridge between operator evolution, path integrals, and observables. They require a systematic revision rather than isolated typographical fixes. Once the points below are addressed, I expect the manuscript could become a valuable review or lecture-note contribution.

Major comments

1. **The SK propagator conventions are internally inconsistent.**

   Section 4.2 defines the statistical function in equation (4.38) as

$$G_K = \langle\!\langle x_r x_r \rangle\!\rangle = \tfrac{1}{2}\langle\{\hat{x}, \hat{x}'\}\rangle,$$

   and defines the retarded function in equation (4.39) as the step function times the commutator, without the conventional factor $-\mathrm{i}$. With these definitions, G_K is real for Hermitian fields, while G_R carries the phase of the commutator. For a Gaussian path integral weighted by $e^{\mathrm{i}S}$, the raw two-point function is i times the inverse quadratic kernel.

Equations (5.93)–(5.95) instead use

$$G_R = \mathcal{D}_R^{-1}, \qquad G_K = -\mathrm{i}\mathcal{N}\,(\mathcal{D}_R\mathcal{D}_A)^{-1}.$$

These are compatible with a convention in which the propagator matrix contains $-\mathrm{i}$ times the raw correlators, but not with equations (4.38)–(4.39). The conflict is visible without any detailed calculation: equation (5.95) makes G_K purely imaginary for a real non-negative noise kernel, whereas equation (4.38) defines it as a real symmetrized expectation value. Appendix A encodes the same mismatch in equations (A.11)–(A.20), where the manuscript sets $\mathbb{KG} = 1$ rather than including the Gaussian factor appropriate to the earlier raw-correlator definition. Section 7 then mixes the choices again: equation (7.92) defines G_K as the real anticommutator, while equation (7.93) includes $-\mathrm{i}$ in the retarded definition.

The author should select one convention and apply it globally. A standard choice is to define the retarded and Keldysh propagators with explicit factors of $-\mathrm{i}$; another is to retain the raw operator correlators used in Section 4. Either is acceptable, but the quadratic inversion, the $+/-$ conversion formulas, the Feynman rules, and the extraction of the real power spectrum must then all be changed consistently. Because these conventions feed into equations (7.76)–(7.87), the author should recheck every factor of i and every overall sign in the inflationary two-point and three-point rules after making this correction.

2. **The Keldysh solution omits the initial-state contribution.**

   Equation (5.95), and more importantly its cosmological counterpart (7.76), gives only the noise-sourced particular solution for the statistical propagator. At a finite initial time the general solution has the schematic form

   $$G_K(t,t') = G_K^{\mathrm{init}}(t,t') + G_R \circ \mathcal{N} \circ G_A,$$

   where G_K^{init} is the initial covariance propagated by the homogeneous retarded and advanced evolution. In a path integral this information is carried by the initial density matrix, equivalently by a boundary action or an $\mathrm{i}\epsilon$ prescription. It is not contained in the bulk noise kernel.

   This omission is exposed by the unitary limit. The text correctly observes around equations (7.61)–(7.66) that the bulk action contains the unitary EFT when the dissipative and noise coefficients are removed. Yet equation (7.76) then gives $G_K = 0$, while the Bunch–Davies state has a nonzero two-point function. The missing Bunch–Davies covariance is precisely the homogeneous or boundary-state term. Equations (7.84)–(7.87) may consistently describe the additive contribution induced by β_1 – the sentence before equation (7.84) briefly says as much – but they are not the general power spectrum of the quadratic theory.

   The statement following equation (7.87) that initial conditions are quickly forgotten also needs hypotheses. It can hold in a late-time attractor regime with positive damping, sufficiently long evolution, and noise dominance. It is not true in the unitary limit, is non-uniform as $\gamma \to 0$, and is especially delicate for constant super-Hubble or zero-momentum modes. Indeed, the massless dissipative example itself contains a non-decaying zero mode in equation (5.106). The revised text should display the general initial-plus-noise solution, specify the limit in which the initial term decays, and state clearly whether equations (7.84)–(7.87) are the full spectrum or only a noise-generated component.

   There is also a measure issue in equation (7.76). The quadratic action (7.66) is written with d^4x, and $\widehat{D}_K$ in equation (7.70) already contains the powers of the scale factor, including the $a^4\beta_1$ term. The convolution should therefore use the same flat coordinate measure d^4z.

Inserting an additional $\sqrt{-g}$ double-counts the scale factor unless $\widehat{D}_K$ is redefined. The powers appearing in equations (7.79)–(7.81) indicate that the subsequent calculation actually used only one such factor. The notation and derivation should be made mutually consistent.

3. **Several algebraic errors affect the inflationary propagator and bispectrum.**

   The cosmological application contains a cluster of independent normalization and scale-factor errors that should be corrected before its phenomenological conclusions are assessed. First, differentiating the Bunch–Davies mode (7.29) gives

$$\pi_k'(\tau) = +\frac{\mathrm{i}H}{\sqrt{4\epsilon c_s k^3}\,M_{\mathrm{Pl}}} c_s^2 k^2 \tau\, e^{-\mathrm{i}c_s k\tau},$$

   whereas equation (7.31) has the opposite sign. Second, the conformal-time interaction integrals (7.35) and (7.39) omit a factor of $a(\tau)$. Indeed, both $a^3\dot{\pi}^3\,dt$ and $a^3\dot{\pi}(\partial_i\pi)^2 a^{-2}\,dt$ become expressions proportional to $a(\tau)d\tau$, since $dt = a\,d\tau$ and $\dot{\pi} = \pi'/a$. The statement that all scale factors cancel is therefore incorrect. The displayed time integrals do not lead to the shapes subsequently quoted in equations (7.37) and (7.42); those shapes should be rederived from a consistently converted interaction Hamiltonian, with special care for the time-derivative interaction.

   Third, direct variation of equation (7.66), using $H = a'/a^2$ and constant γ, gives, up to the manuscript's overall kernel convention,

$$\widehat{D}_R = a^2\partial_\eta^2 + a^3(2H+\gamma)\partial_\eta - a^2 c_s^2\nabla^2,$$

   and its adjoint contains $a^3(2H-\gamma)\partial_\eta - 3a^4H\gamma$. Equations (7.68)–(7.70) instead contain $2H/a$ in the first-derivative coefficients and $-3a^3H\gamma$ in the advanced kernel. These powers of a are incompatible both with the action and with the differential equation (7.73), which in fact has the friction coefficient expected from the corrected retarded operator. Finally, the Bessel-function Green function in equation (7.75) fails the delta-function jump condition: the Wronskian $J_\nu Y_\nu' - J_\nu' Y_\nu = 2/(\pi z)$ fixes the magnitude of the prefactor to $\pi/2$, not π^2 (the overall sign depends on the declared ordering and kernel convention). The author should redo the jump normalization explicitly and propagate the correction through equations (7.76)–(7.87).

   These are not merely presentational points: the missing scale factor changes the time integrals and momentum dependence of the bispectrum, while the kernel and Wronskian errors change the normalization of the retarded response and hence of the noise-sourced spectrum. A revised manuscript should show at least one explicit mode-function differentiation, one conformal-time conversion, and the Green-function jump check so that the final correlators can be audited.

4. **The local SK and influence-functional examples need a consistent derivation.**

   The phase-space treatment correctly derives the complete-positivity condition (5.30), which in particular requires both canonical diffusion directions when the friction κ is nonzero. The later scalar action (5.87), however, retains a friction term $\gamma\phi_a\dot{\phi}_r$ with only a single ϕ_a^2 noise structure. Read as a Markovian quantum dynamical map, this corresponds to setting one diagonal diffusion coefficient to zero and therefore cannot satisfy (5.30) for $\gamma \neq 0$. Integrating out momentum in the first-order action (5.12) also shows why: a general CP phase-space noise matrix generates not only x_a^2 but also $\dot{x}_a^2$ and mixed/contact structures. The author should either restore the missing diffusion terms and state their positivity bound, or label (5.87) explicitly as a semiclassical Caldeira–Leggett-type truncation that is not by itself a CP quantum evolution. Positivity of the single kernel $\mathcal{N}_{\mathbf{k}}$ is not sufficient to make the reduced map completely positive.

There is a related sign inconsistency. Eliminating p_r from (5.12) imposes $p_a = \dot{x}_a - \kappa x_a$ and yields, after integration by parts,

$$-\int dt\, x_a(\ddot{x}_r + \kappa\dot{x}_r + \Omega^2 x_r)$$

up to endpoints. Equations (5.57) and (5.87) use the opposite sign. The zero of the classical equation is unaffected, but the sign controls the response-source convention and must agree with the retarded kernel and Hubbard–Stratonovich transformation. In fact, the $+\mathrm{i}\int \xi x_a$ choice in (5.52) combines with (5.51) to produce $\mathcal{E} + \xi$, not the $\mathcal{E} - \xi$ written in (5.53), unless one simultaneously changes $\xi \to -\xi$. These choices should be fixed once, including endpoint/contact terms, and followed through the propagator dictionary.

Two earlier influence-functional steps also need repair. In equations (4.61)–(4.63), the environment wavefunctionals should be evolved with $S_E + S_{\text{int}}$ for a prescribed system history. Using the full S_{tot} there re-inserts the system action even though it was already factored out in (4.55), and therefore double-counts it. Moreover, the step from (4.79) to (4.80) is exact in the displayed quadratic form only for a Gaussian initial environment state; a nonzero bath one-point function supplies a linear term, and non-Gaussian states supply higher connected cumulants even when the bath action is quadratic. The assumptions should be stated and the formula written in terms of connected correlators. Finally, the composition argument (5.4)–(5.7) should either restrict the local action to first derivatives or glue the additional derivative/phase-space boundary data required by higher-derivative terms.

5. **The Gaussian Lindblad example violates its own complete-positivity bound.**

   Equation (3.144) is followed by the condition

$$D_{pp}D_{xx} - D_{xp}^2 \geq \frac{\kappa^2}{16} \tag{3.145}$$

   in the conventions of the manuscript. Equation (3.146) then assigns pure damping

$$D_{xx} = \frac{\kappa}{8\omega}, \qquad D_{pp} = \frac{\kappa\omega}{8}, \qquad D_{xp} = 0$$

   and says that this saturates the bound. In fact, these values give $D_{pp}D_{xx} = \kappa^2/64$, which violates (3.145). Equation (3.147) has the same factor-of-two error in each diffusion coefficient and violates the bound for $N_{\text{th}} < 1/2$, including the zero-temperature limit that is supposed to be a valid amplitude-damping channel.

   With the canonical normalization used here, the damping channel $L = \sqrt{\kappa}\, a$ gives

$$D_{xx} = \frac{\kappa}{4\omega}, \qquad D_{pp} = \frac{\kappa\omega}{4}, \qquad D_{xp} = 0,$$

   and the thermal values acquire the same replacement $1/8 \to 1/4$. This agrees with the phase-space Kossakowski formulation in equations (5.73)–(5.75). The footnote attached to equation (3.144) should also be corrected: friction is not generated by Hermitian jump operators alone. It arises from the imaginary antisymmetric part of the Kossakowski matrix, or equivalently from non-Hermitian linear jumps such as a. This is an important example because the notes later use precisely this noise-versus-friction inequality to motivate complete positivity in effective actions.

6. **Two structural statements about Lindblad maps need correction.**

   First, the separate jump-operator redundancies in equations (2.87)–(2.89) are correct, but their combined form in equations (2.90)–(2.91) is not. If

$$K_\mu = \sum_\nu U_{\mu\nu} L_\nu, \qquad L'_\mu = K_\mu + \alpha_\mu \mathbf{1},$$

then the compensating Hamiltonian is

$$H' = H - \frac{\mathrm{i}}{2}\sum_{\mu}(\alpha_\mu^* K_\mu - \alpha_\mu K_\mu^\dagger) + c\mathbf{1}.$$

Equation (2.91) instead contracts the post-rotation coefficients α_μ with the unrotated L_μ. It is not invariant for a generic U and α. One may either write the correction in terms of K_μ or explicitly rotate the shift coefficients back to the original basis.

Second, differentiability by itself does not justify equations (2.145)–(2.146). To define a time-local generator from a one-sided map one normally uses $\mathcal{L}_t = \dot{\Phi}(t,t_0)\Phi(t,t_0)^{-1}$, which requires invertibility (or a more careful image/kernel construction). Without a composition law, the result can also retain dependence on the chosen t_0. Conversely, legitimate CPTP evolutions can become non-invertible at finite time, whereas an ordinary finite time-ordered exponential is invertible. The later CP-divisibility statement does mention invertibility, but the construction preceding it should state the same assumptions and discuss what changes at singular times. The analogous discussion in Section 5 is better motivated because the cut-open local kernel supplies intermediate maps, and it would be useful to align the two presentations.

7. **Several foundational claims in Sections 1–2 require sharper mathematical scope.**

   These notes are aimed at students, so the statements that establish the basic formalism should be exact.

   - The paragraph after equation (1.98) uses "product" and "separable" interchangeably and then states that both product and entangled states are pure. This is correct only for pure state vectors. Mixed separable states are convex combinations of product states, and mixed entangled states also exist. The terminology should be repaired before the later discussion of mixedness and decoherence.
   - Equation (1.180), the concavity of von Neumann entropy, is correct, but equations (1.181)–(1.184) do not prove it for non-commuting density matrices. Diagonalizing one matrix and invoking scalar Jensen is insufficient. The author should either provide a valid operator proof or cite the theorem and present the scalar discussion only as intuition. The same part of the notes would benefit from stating the support convention for relative entropy and citing Klein's inequality.
   - The discussion surrounding equations (1.202), (1.209)–(1.213), and (1.216) identifies mixedness too quickly with decoherence and then with a classical statistical ensemble. Repeating the model also assumes a fresh, uncorrelated, reset apparatus at each step. The reduced state may be operationally diagonal for the chosen subsystem while phase information remains in system–environment correlations; recoherence, the distinction between proper and improper mixtures, and the measurement-outcome problem are not thereby removed. This should be stated explicitly, without weakening the useful demonstration of local suppression of interference.
   - The manuscript moves between finite and infinite dimensions without always flagging the change. In infinite dimension, density operators are trace-class and do not form a norm-compact subset of $\mathcal{B}(\mathcal{H})$; Schmidt decompositions and entropy statements require hypotheses; and unbounded GKSL generators have important domain qualifications. A clean solution would be to declare Sections 1–2 finite-dimensional unless otherwise stated, then identify explicitly when the oscillator and field-theory examples require extensions. Purification uniqueness should similarly be stated as unitary freedom on a common sufficiently large ancilla, and as an isometry when ancilla spaces differ.

8. **The approximation status of the cosmological applications should be stated more precisely.**

   The leading stochastic-inflation equations are useful and, within their intended approximation, the algebra is largely correct. However, some phrases make them sound more general than the derivation. The replacement of the operator noise by a classical Gaussian variable after equation (8.50) uses a sharp window, $\epsilon \ll 1$, free Bunch–Davies crossing modes, leading gradient order, and leading infrared classicalization. The neglect of instantaneous long–short interactions is a leading-log or leading-stochastic statement, not a general property of an interacting open theory. Similarly, the "exact" free variances in equations (8.27)–(8.28) are exact within the stated leading super-Hubble coarse-graining approximation and chosen cutoff/window, not exact renormalized local QFT variances. These domains should accompany the headline claims.

   The same care is needed when the open EFT imposes the thermal relation $\beta_1 = 2\pi\gamma T$ before equation (7.87). Since the notes otherwise avoid specializing to equilibrium, the author should state the Fourier and KMS conventions and the regime in which this local fluctuation–dissipation relation applies. More broadly, the necessary SK inequality $\operatorname{Im} S_{\text{eff}} \geq 0$ should not be described as if it were sufficient for positivity or complete positivity of the reduced map; the later and very useful comparison with the stronger Kossakowski constraint should be signposted at the first occurrence.

   The terminology around equation (6.24) should also be corrected. A momentum-space spectrum proportional to k^{-3} is scale invariant for a dimension-zero perturbation in the sense that $k^3 P(k)$ is constant. Its real-space two-point function is not independent of separation: the Fourier transform is logarithmic and requires infrared and ultraviolet prescriptions. Stating the dimensionless-spectrum criterion avoids a misleading claim and makes clear which variable, especially ζ, is being discussed.

9. **The novelty and literature positioning should reflect the genre of the manuscript.**

   As lecture notes, the work can be publishable without a new theorem, but the reader should be able to distinguish textbook material, the author's synthesis, and results imported from recent research papers. The present references-and-resources subsection is helpful, and the captions identify reused figures, but the advanced Sections 5 and 7 would benefit from a more explicit roadmap. In particular, the jump-operator positivity-improvement prescription, the Gaussian CP bound, the open-EFT operator basis, and the power-spectrum and bispectrum formulas should each be accompanied at the point of use by a clear statement of whether the result is reviewed, rederived, generalized, or newly proposed here. The integrated operator/SK/cosmology perspective is itself interesting; presenting its provenance precisely will strengthen rather than diminish the manuscript's contribution.

Minor comments

1. In the short-time Kraus sketch, equation (2.74) with $M_\mu = \sqrt{\delta t}\, L_\mu + O(\delta t)$ generically produces cross terms of order $\delta t^{3/2}$, not an $O(\delta t^2)$ remainder as written in equation (2.75). A more careful small-time Kraus gauge, or an $o(\delta t)$ statement at the channel level, would avoid this mismatch.

2. Equation (2.55) should say that a *normalized* Kraus representation is equivalent to CPTP evolution. An unnormalized Kraus form guarantees complete positivity but need not preserve the trace.

3. The interaction Hamiltonian in equation (2.106) should be stated to be Hermitian, either by taking the S_α and X_α Hermitian or by including conjugate-paired terms. The conditions ensuring that the Lamb-shift expression (2.133) is Hermitian should likewise be stated. Near equation (2.126), secularization separates fixed Bohr-frequency sectors, but populations and coherences need not fully decouple when energy levels or Bohr frequencies are degenerate.

4. The first equality in equation (5.21) is off by a factor of two. Since the Jacobi identity and $[x, p] = \mathrm{i}$ give $[x, [p, \rho]] = [p, [x, \rho]]$, the term $-N_{xp}[x, [p, \rho]]$ equals $-\frac{N_{xp}}{2}([x, [p, \rho]] + [p, [x, \rho]])$, as used later in equation (5.24), rather than the extra one-half multiplying the single commutator in (5.21).

5. The Gaussian solution following the diffusion equation (8.59) is the Euclidean free-particle heat kernel, not the ground state of the quantum harmonic oscillator as stated in the footnote attached to that discussion.

6. The manuscript would benefit from a final copy-edit. Representative examples include "non-decreae" in the unitality heading, "An well studied example" in Section 9, "my understand" in the references discussion, and "what has been studies" in the final paragraph. These do not affect the physics but are numerous enough to interrupt an otherwise polished presentation.

Recommendation

I recommend major revision. The notes have a strong pedagogical premise, a useful cross-disciplinary scope, and several well-chosen derivations. The principal obstacles are identifiable and repairable: the author should make the SK two-point conventions consistent, restore the initial-state contribution to the statistical propagator, and rederive the cosmological kernels, Green-function normalization, and bispectra with the correct measures and scale factors. The local scalar example should then be reconciled with the complete-positivity criterion, and the Gaussian CP and Lindblad-map statements corrected. Finally, the mathematical scope and approximation claims should be sharpened. Because the convention and algebraic problems affect the manuscript's central conceptual bridge and its observable power spectra and bispectra, I would want to see a revised manuscript before recommending publication. If these revisions are carried out carefully, the work should be interesting and useful to the intended high-energy theory and cosmology audience.