Referee Report

"Bosonic SPT and invertible phases and its relation to Steenrod's problem"

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Summary and overall assessment

The manuscript studies oriented bosonic invertible and symmetry-protected topological phases through the first two nontrivial rows of the Atiyah–Hirzebruch spectral sequence (AHSS). Its organizing observation is that the first differential linking the ordinary-cohomology row to the row generated by the oriented bordism class in degree four is controlled by the mod-three operation $\beta\mathcal{P}^1$ (or its Pontryagin-dual operation $i\mathcal{P}^1\beta$). This is presented as the bosonic, oriented counterpart of the mod-two Steenrod-square structure familiar from fermionic supercohomology.

Section 2 develops the idea through examples. The usual gravitational Chern–Simons response is reviewed first. The manuscript then derives the integral combination

$$\frac{p_1 c_1 - c_1^3}{3}$$

for a $4+1$-dimensional $U(1)$-symmetric phase, using the mod-three Wu formula in equation (2.28). Pullback to finite groups gives the phases in equation (2.37), the order-nine $\mathbb{Z}_3$ relation in equation (2.43), and the addition law in equation (2.46). The most distinctive example is the class

$$\mathcal{P}^1\beta(x\widetilde{x}) = (\beta x)^3\widetilde{x} - x(\beta\widetilde{x})^3,$$

equation (2.49), whose image in $H^7(B\mathbb{Z}_3^2; U(1))$ is nonzero but whose pairing with every closed oriented manifold vanishes by equation (2.52). Section 3 recasts these examples in the AHSS, treats $K(\mathbb{Z},3)$, $K(\mathbb{Z},4)$, BG_2, and $BSpin$, and proposes a two-stage approximation to bosonic phases. The appendices discuss the order-nine extension using eta invariants and Brown–Peterson theory, compute a bordism group for $K(\mathbb{Z},4)$, and derive the relevant d_5.

There is much to like in this paper. The mod-three viewpoint gives a coherent explanation of several calculations that otherwise look unrelated. The elementary Wu-formula arguments around equations (2.17)–(2.18) and (2.49)–(2.52) are effective, and the relation to Thom's representability problem is conceptually memorable. The single-factor $\mathbb{Z}_3$ extension is supported in three complementary ways: equation (2.43), the AHSS exact sequence (3.37), and the eta/Brown–Peterson checks in equations (A.12) and (A.23c). The comparison between the $K(\mathbb{Z},4)/BF_4$ and BG_2 cases is also potentially useful.

Several central claims are not yet established at the level required for publication, however. Most importantly, the product-group result in equation (3.52) does not follow from the displayed AHSS and is not derived in Appendix A. The asserted identification $P^8(BG) \cong [BG, BSpin(k)]$ in equation (3.84) also replaces a Postnikov truncation by the full classifying space without addressing higher lifting obstructions. In addition, the paper alternates between a cohomology class on arbitrary cycles and a physical invertible phase on manifolds, even though the class at the center of the Steenrod example is zero in the very bordism-based classification of phases adopted in Section 3. These issues affect correctness, scope, and the interpretation of the main results rather than only presentation.

Recommendation: major revision. I regard the basic program as interesting and potentially publishable, but I do not recommend publication in the present form. A revision should prove or appropriately weaken the product-group and $BSpin$ claims, define precisely the physical category in which a class invisible on manifolds is meant to be a phase, and make the Chern–Simons/differential-cohomology refinements and homotopy-theoretic conventions unambiguous.

Correctness and logical structure

The mathematical spine of the paper is credible when phrased at the associated-graded level. The mod-three Wu relation explains both the corrected $U(1)$ characteristic number and the first nonzero oriented-bordism differential. Appendix C gives a plausible derivation of the Anderson- and Pontryagin-dual operations, and the examples agree with the filtration picture. The paper is strongest when it states a kernel or cokernel in the AHSS and then supplies an independent invariant that resolves the extension, as it does for $\widetilde{\Omega}_5^{SO}(B\mathbb{Z}_3)$.

The logical problems arise when the manuscript moves beyond that associated-graded information. Equation (3.51) fixes the order of the product-group answer but not its abstract group structure. An equivalence through a finite Postnikov stage does not control maps from an infinite-dimensional BG into the full space. Likewise, the fact that a cohomology class vanishes on all manifolds establishes that it lies in the kernel of the edge map; it does not by itself construct a new physical phase on singular lattices. These three transitions need either proofs or explicit qualifications.

The paper would benefit substantially from promoting its main statements to propositions. At present, proved classifications, reviewed facts, choices of notation, and prospective constructions all appear in continuous prose. Formal statements with hypotheses would make clear which symmetry types are included, whether groups are reduced or unreduced, which backgrounds are topological or differential, and which results hold only at the two-stage level.

Novelty and significance

The principal novelty is a synthesis rather than the discovery of every individual group. The gravitational Chern–Simons example is standard; the $4+1$-dimensional $\mathbb{Z}_3$ beyond-cohomology phase and its order-nine structure were treated by Yang and Cheng; the $K(\mathbb{Z}, 3)$ bordism calculation is attributed to the recent work of Jia, Wang, and Zhang; and the nonrepresentable seven-cycle is Thom's classical example. The recent paper by Wan, Wang, and Yau also overlaps with the mod-three vanishing of a Pontryagin-class expression. These works are cited, but the manuscript does not yet separate rederivation, synthesis, and genuinely new classification sharply enough.

The apparently new contributions include the systematic identification of the mod-three two-row structure, the general finite-group addition law, the explicit physical reading of Thom's class, the G_2–F_4 comparison, the full product-group classification in equation (3.52), and the proposed $BSpin$ realization. The last two are also the least well established parts of the manuscript. Their proofs therefore matter directly to the paper's claim of research novelty. If corrected, the overall package should be of interest to researchers working on generalized symmetries, invertible field theories, anomalies, and topological phases. If those claims are weakened, the authors should explain what new theorem remains beyond a useful pedagogical organization of known calculations.

Major comments

1. **The extension leading to equation (3.52) is not resolved in the manuscript.**

   The AHSS calculation gives equation (3.51),

   $$0 \longrightarrow (\mathbb{Z}_3)^4 \longrightarrow \mathrm{Hom}\big(\Omega_7^{SO}(B\mathbb{Z}_3 \times B\mathbb{Z}_3), U(1)\big) \longrightarrow (\mathbb{Z}_3)^3 \longrightarrow 0.$$

   This determines the order of the middle group but does not determine whether the extension is $\mathbb{Z}_9^2 \oplus \mathbb{Z}_3^3$, $\mathbb{Z}_9^3 \oplus \mathbb{Z}_3$, or another allowed group of the same order. The text says that Appendix A.3 resolves the problem, but Appendix A.3 computes $BP_*(B\mathbb{Z}_3)$, not $BP_*(B\mathbb{Z}_3 \times B\mathbb{Z}_3)$. Relation (A.23d) shows how the classes pulled back from either cyclic factor participate in nonsplit extensions. It does not compute the mixed smash-product contribution, identify all extension maps, or show that the remaining mixed class has order three.

   This is a central advertised classification and requires a real calculation. The authors could provide a Brown–Peterson Kunneth or Tor spectral sequence for the product, an explicit set of bordism generators and relations, or enough independent eta/rho invariants to determine their orders. The proof should identify which filtration representatives form the two $\mathbb{Z}_9$ factors and why no third nonsplit extension or higher-order element occurs. It should also justify the three generators displayed only "suggestively" in equation (3.52); expressions involving division by three are not definitions of $U(1)$-valued cohomology classes without a specified refinement. Until this calculation is supplied, equation (3.52) and the claims that depend on it should be presented as conjectural.

2. **The $BSpin$ universality claim confuses a two-stage truncation with the full classifying space.**

   The argument in Section 3.3.1 plausibly identifies the two-stage Postnikov space P_8, with nonzero homotopy groups in degrees four and eight, with $\tau_{\leq 8} BSpin(k)$ for stable k. This does not imply

   $$P^8(BG) \cong [BG, BSpin(k)]$$

   as written in equation (3.84). For an infinite-dimensional source such as BG, a map to $\tau_{\leq 8} BSpin(k)$ need not lift to the full $BSpin(k)$; higher Postnikov invariants give further obstruction classes, and different lifts can carry higher characteristic data. An equivalence through degree eight only controls maps from suitably low-dimensional complexes, not arbitrary BG.

   The manuscript itself recognizes part of the problem immediately after equations (3.85)–(3.86), where it states that no construction of a map $g : BG \to BSpin(k)$ has been given for a general kernel class. That acknowledgment is incompatible with the earlier statement that every two-stage phase is the pullback of a $BSpin(k)$-symmetric phase. The correct general expression is

   $$P^8(BG) = [BG, P_8] \simeq [BG, \tau_{\leq 8} BSpin(k)],$$

   with an actual $Spin(k)$ bundle available only when the relevant map lifts. The authors should either work consistently with the Postnikov truncation, restrict to an eight-skeleton or another finite-dimensional range, or calculate the higher lifting obstructions. The Chern–Simons construction in equations (3.77)–(3.83) is valid for a chosen lift g, but it does not establish the advertised universal realization for all two-stage classes.

3. **A cohomology class that vanishes on all manifolds is not automatically a distinct physical phase.**

   Equations (2.49)–(2.52) establish a clean mathematical fact: a nonzero element of $H^7(B\mathbb{Z}_3^2; U(1))$ lies in the kernel of the map to $\mathrm{Hom}(\Omega_7^{SO}(B\mathbb{Z}_3^2), U(1))$. In the bordism-based definition adopted in equations (3.3), (3.11), and (3.12), this means that the corresponding Dijkgraaf–Witten action is the trivial oriented invertible phase. Its nonzero pairing with an abstract homology cycle shows that the cycle is not representable by an oriented manifold; it does not by itself produce a nontrivial partition function on the physical spacetimes included in that classification.

   The proposed interpretation in Section 2.5 as a phase that detects the "non-manifold-ness" of a lattice needs additional structure. A general simplicial complex need not have an oriented fundamental class, local manifold links, reflection positivity, or a class of Pachner moves under which a state sum should be invariant. The authors should specify whether they mean oriented pseudomanifolds, stratified spaces, arbitrary cycles with chosen fundamental chains, or a concrete Hamiltonian lattice construction. They should then define the partition function and its equivalence relation in that category. Without such a construction, the established result should be described as a nonzero group-cohomology class invisible to oriented manifolds, while the condensed-matter interpretation should be clearly labeled as a proposal. This distinction is central to the title and stated significance of the paper.

   There is a related gap in the finite-dimensional model used after equation (2.52). The statement that $S^7/\mathbb{Z}_3$ has the same low-degree cohomology as $B\mathbb{Z}_3$ is not sufficient in the top degree relevant to the seven-dimensional class; truncation changes top cohomology. The authors should compute the restriction of equation (2.50) to $(S^7/\mathbb{Z}_3)^2$ directly and show that it remains nonzero, for example through its integral Bockstein in degree eight.

4. **The continuum actions need a precise differential-cohomology or transgression definition.**

   After equation (3.41), the $5+1$-dimensional $K(\mathbb{Z}, 3)$ phase is said to be "simply" $\int_{M_7} H \wedge p_1$. As written, this is dimensionally an integral on a seven-manifold rather than an action on the physical six-manifold. If M_7 is closed and H and p_1 are integral classes, the integral is an integer and its exponential with the convention of Section 2 is trivial. The nontrivial six-dimensional response resides in a Chern–Simons transgression: one should start with a physical manifold M_6, a differential cocycle refining the two-form background, and a bounding extension W_7, then prove independence of the extension modulo integers. Torsion backgrounds and large gauge transformations should be included.

   The same issue is present, though handled more concretely, in equations (2.37) and (3.79). Symbols such as $p_1/3$, $x/3$, and the divided generators in equation (3.52) are not ordinary integral cohomology classes. Depending on context they denote an integral characteristic number on oriented bordism, a differential-cohomology Chern–Simons refinement, or a chosen lift. These meanings should not be interchanged. A short subsection fixing the differential-cohomology conventions would make the physical actions well-defined and would clarify exactly what is being classified by the Anderson dual as opposed to the Pontryagin dual.

5. **The finite-group completeness claim must distinguish arbitrary Dijkgraaf–Witten terms, character pullbacks, and the reduced classification.**

Equation (2.47) gives the expected two-row extension of $H^1(BG;U(1))$ by $H^5(BG;U(1))$, and equation (2.46) supplies a useful cocycle for that extension. However, the statement in Section 4 that the two $U(1)$-symmetric Chern–Simons terms can be used to produce "all finite-group symmetric phases" is not correct as written. Pulling back c_1^3 along one-dimensional characters produces special classes of the form $x(\beta x)^2$; an arbitrary element of $H^5(BG;U(1))$ need not be generated in this way. The construction of the full extension also includes the independently specified Dijkgraaf–Witten term $D(\alpha)$ of equation (2.32).

The word "all" also silently omits the purely gravitational $\mathbb{Z}_2$ contribution from $\Omega_5^{SO}(\mathrm{pt})$, which is restored in Appendix A, and assumes an internal unitary symmetry with oriented spacetime. The authors should formulate a proposition for the reduced, symmetry-dependent group, derive equation (2.47) from the AHSS for general finite G, and state explicitly why no other differential contributes in total degree five. The proposition should distinguish arbitrary $D(\alpha)$ from the pullbacks used to construct $C(x)$, and should state how changing choices of Chern–Simons refinements changes the displayed extension cocycle by a coboundary. Antiunitary, orientation-reversing, and crystalline symmetries are outside the analysis and should be excluded explicitly.

6. **The generalized signature formula and the normalization arguments need to be made auditable.**

   Equation (3.27) is essential to the claimed generators in equations (3.33), (3.60)–(3.66), and (3.70)–(3.72). The unusual factors 2^j are precisely what produce the required divisibility statements. The manuscript should define the elliptic operator whose index is denoted by $\sigma_E(M)$, the structure and connection assumed on E, and the Chern-character convention, and should match the formula explicitly to the cited theorem. This is particularly important in dimensions six and eight, where readers may otherwise confuse the construction with the more familiar signature operator twisted by a flat local system.

   The short arithmetic steps that extract a class divided by three should also be stated. For example, the integrality of $8(c_2^2 - c_2p_1)/3$, together with the ordinary integrality of $c_2^2 - c_2p_1$, implies the desired divisibility because 8 is invertible modulo three. Analogous reasoning is used for the $BSpin$ generator. Writing these steps makes clear that no division of an arbitrary integral cohomology class is being assumed. Finally, the spectral-sequence diagrams should indicate their axes, coefficient rows, reduced/unreduced convention, and all differentials relevant to the displayed total degree; the current stars and unlabeled diagonals make several assertions difficult to check.

7. **Appendix A must fix its eta-invariant and Brown–Peterson conventions.**

   Equation (A.12) is used as a bordism pairing of denominator nine. The raw eta invariant of a twisted odd signature operator is not, without qualification, an $\mathbb{R}/\mathbb{Z}$-valued bordism invariant. The relevant object should be stated as the reduced eta or rho invariant, with the trivial representation subtraction, normalization, and reduction modulo integers made explicit. The authors should explain why the expression in equation (A.12) is exactly that reduced pairing on the reduced bordism class. This would turn a plausible and useful check into a reproducible proof.

   Equation (A.22) also writes the p-series of BP as an ordinary sum $[p](x) = px + v_1x^p + v_2x^{p^2} + \cdots$. In a p-typical coordinate the standard statement uses the formal group law, $[p]_F(x) = px +_F v_1x^p +_F v_2x^{p^2} +_F \cdots$, and the coefficients in an ordinary power-series expansion depend on the coordinate and generator convention. The authors should specify whether the v_i are Araki, Hazewinkel, or rescaled generators and derive the low-degree relations (A.23a)–(A.23d), allowing for units where appropriate. The nonvanishing of $v_1 z_1$ and $v_1 z_2$ should be stated rather than inferred only from the desired extension. These conventions are also necessary before the Brown–Peterson method can be extended to the product calculation required for equation (3.52).

There is also an indexing inconsistency before equation (A.22). Equation (A.19) writes $[p](x) = \sum_{i\geq 0} a_i x^i$ and then says $a_0 = p$. A formal-group p-series has zero constant term and linear coefficient p. The intended expression must instead use x^{i+1}, or set $a_1 = p$ and shift the indices. Since the recurrence in equation (A.21) and the relations (A.23a)–(A.23d) depend on this indexing, those formulas should be rederived after the convention is corrected.

8. **The proposed bosonic supercohomology analogue is heuristic, and the precise obstruction should replace the formal manipulations.**

   Section 3.4 is commendably candid that a cochain-level construction has not been achieved. Nevertheless, the manipulations following equation (3.103) risk suggesting more than has been shown. A set-theoretic section denoted by $b/3$ is not natural or additive, and the fact that $\mathcal{P}^1\beta b - p_1\beta b$ pairs trivially with closed oriented manifolds does not by itself imply the existence of a universal natural cochain $\zeta(b)$ with the proposed coboundary. Such a ζ is a cochain homotopy compatible with tangential data and gauge equivalence, which is a stronger statement than vanishing of characteristic numbers.

   The authors should formulate the missing object as a precise obstruction or lifting problem, state the required naturality and equivalence conditions, and separate conjectural formulas from established two-stage homotopy data. If the construction is retained only as an outlook, this limitation should also be reflected in the introduction so that the analogy with Gu–Wen supercohomology is not read as a completed classification.

9. **The novelty claims and the proof status of the principal results should be separated explicitly.**

   The introduction should contain a concise list of new results. In particular, it should say which part of the order-nine $\mathbb{Z}_3$ phase goes beyond Yang–Cheng, which part of the $K(\mathbb{Z}, 3)$ analysis goes beyond the recent Jia–Wang–Zhang calculation, and how the discussion of p_1 differs from the recent Wan–Wang–Yau work. The statement near the beginning that the phenomenon first manifests in $5+1$ dimensions is juxtaposed with a citation whose subject is a $4+1$-dimensional phase and with the manuscript's own $4+1$-dimensional order-nine example; the intended meaning of "first non-triviality" must be specified.

   I recommend stating at least four formal results: the reduced $4+1$-dimensional finite-G extension; the precise kernel element associated with Thom's example; the product-group calculation, once proved; and the correct Postnikov-level $BSpin$ statement. Each should list its hypotheses and point to a complete proof. This would also make the paper's genuine advance much easier to assess.

Minor comments

1. The title should read "Bosonic SPT and invertible phases and *their* relation to Steenrod's problem," or be recast in the singular.

2. In Section 2.5, the notation for $H^8(B\mathbb{Z}_3 \times B\mathbb{Z}_3; \mathbb{Z})$ should display $(\mathbb{Z}_3)^5$, not a single $\mathbb{Z}_3$ followed by five apparent generators.

3. Section 3.3.1 uses $k \geq 9$, while Section 4 says $k > 9$. The stable range should be stated consistently and justified.

4. Equation (A.1) is written with $MTSO$, whereas the remainder of the calculation uses MSO. These spectra are not interchangeable by notation alone; the intended Thom spectrum and any shift or equivalence should be specified.

5. The line following equation (A.23c) should derive $9z_3 = -v_1(3z_1) = 0$, not the ambiguous product of extra factors of three currently printed.

6. The boundary notation in Section 2.2 should clarify that the reduced bordism compares (M_5, f) with a second, oppositely oriented copy carrying the trivial bundle. Writing the same symbol M_5 on both boundary components obscures the construction.

7. Several spectral-sequence displays contain only stars. A legend and total-degree lines would make equations (3.8), (3.11), (3.12), (3.42), and (3.67) much more useful.

8. There are recurrent grammatical and typographical problems, including "Steenrod more than one problems," "would then has," subject–verb disagreement for "differentials," duplicated punctuation, and inconsistent use of "Dijkgraaf–Witten phase" versus "cohomology class." A careful copy edit is needed.

9. The introduction should formulate Steenrod's problem in terms of a homology class not representable as $f_*[M]$. For a general target space X, speaking of the "Poincaré dual" of a class in $H^*(X; \mathbb{Z})$ is not appropriate and risks conflating realization by a map with realization by an embedded submanifold.

10. In Appendix A.3, the sentence following equation (A.21) should say that $E_*(B\mathbb{Z}_p)$, not $E_*(\mathbb{CP}^\infty)$, is generated over E_* by the classes z_i with the displayed relations.

Recommendation

The manuscript contains an appealing organizing idea and several convincing calculations, but two of its principal new claims—the full product-group extension and the universal $BSpin$ realization—are not proved as stated. The physical status of the class that is invisible on manifolds also needs a sharper definition. I therefore recommend **major revision**. A revised paper that supplies the missing product calculation, replaces equation (3.84) by a correct Postnikov-level statement or proves the required lifts, and carefully separates bordism phases from invariants of more general cycles would merit serious reconsideration.