Referee Report

on "Casimir effect for a massive scalar field confined between parallel plates with a spatially varying effective mass"

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Summary

The manuscript considers a real scalar field in Minkowski space with the static quadratic profile $m^2_{\text{eff}}(\rho) = m^2 + \alpha^2\rho^2$ and imposes Dirichlet conditions at two planes separated by a distance L. The transverse problem is treated as a two-dimensional harmonic oscillator. The authors then decompose the mode sum into an $l = 0$ part, called the Landau-like contribution, and an $l \neq 0$ part, called an additional contribution. After Poisson resummation in the direction normal to the plates, the first two Poisson terms are subtracted and the remaining term is interpreted as the renormalized Casimir energy. Weak- and strong-coupling asymptotics and numerical plots are presented.

The underlying action defines a sensible static trapped-field spectral problem, and Poisson resummation along the finite interval is an appropriate way to isolate a plate-interaction term. However, the transverse eigenfunctions and eigenvalues are incorrect, the mode count imports a Landau degeneracy that the system does not possess, and the proposed Landau-like/additional decomposition has no invariant physical meaning. There are also independent errors in both asymptotic expansions. These problems affect every quantitative result from equation (11) onward, including the normalization of the energy and Figs. 2 and 3.

Recommendation

I recommend **rejection in the present form**. The difficulties are not localized mistakes that can be repaired while retaining the central calculation: the geometry and observable must first be defined consistently, the spectrum and state counting must be replaced, and the regularized energy, asymptotics, numerics, and physical interpretation must all be recomputed. A fundamentally reconstructed analysis of the trapped scalar problem could be worth considering as a new submission, but the present manuscript does not yet establish correct, novel, and publication-ready results at the level expected for a high-energy theory journal.

Major comments

1. **The transverse oscillator eigenproblem in equations (7)–(11) is solved incorrectly.**
   After separation of the t, z, ϕ dependence, the transverse radial operator is

$$\mathcal{H}_{\perp,l} = -\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right) + \frac{l^2}{\rho^2} + \alpha^2\rho^2.$$

With $x = \alpha\rho^2$, its regular, normalizable eigenfunctions are proportional to

$$\rho^{|l|} e^{-x/2} {}_1F_1(-n; |l|+1; x), \qquad n = 0, 1, 2, \ldots,$$

and the transverse eigenvalue is

$$\lambda_{nl} = 2\alpha(2n + |l| + 1).$$

The manuscript instead uses the hypergeometric argument $\alpha\rho^2/2$ and obtains $\lambda_{nl} = \alpha(2n + |l| + 1)$. A direct ground-state check exposes the factor of two: applying $\mathcal{H}_{\perp,0}$ to $e^{-\alpha\rho^2/2}$ gives $2\alpha e^{-\alpha\rho^2/2}$, not $\alpha e^{-\alpha\rho^2/2}$.

This error is also visible internally. Equation (15) gives the nonrelativistic oscillator frequency $\omega_{\mathrm{H}} = \alpha/(mc)$. The two-dimensional oscillator levels therefore shift the nonrelativistic energy by $(2n + |l| + 1)\alpha/m$ in natural units, whereas expanding the spectrum in equation (11) gives only half that spacing. Equations (17) and (18), all heat kernels used in Section III, the dimensionless gaps, and the numerical curves must consequently be recomputed.

2. **Equation (25) introduces an unjustified Landau degeneracy and does not define a consistent energy per unit area.**

   The trace in equation (24), with the labels (j, n, l), already counts each normalized oscillator mode once. The additional replacement

$$\frac{V}{(2\pi)^3} \int \mathrm{d}^3k \longrightarrow \frac{A\alpha}{2\pi} \sum_{j,n,l}$$

   is not a continuum-to-discrete density-of-states formula for the stated system. A uniform magnetic field has a macroscopic degeneracy per unit area because of magnetic translations, or equivalently a guiding-center quantum number. The radial profile $\alpha^2\rho^2$ selects an origin and breaks transverse translation symmetry. Its oscillator shells have only the finite degeneracy already contained in (n, l); no additional factor $A\alpha/(2\pi)$ exists.

   This point is tied to the geometry. The Laguerre modes used in Section II assume $0 \le \rho < \infty$. With infinite plates, the transverse trap localizes the modes, so the plate interaction is a total energy associated with that trap rather than an extensive, spatially uniform energy proportional to an arbitrary area A. If the plates instead have finite area as suggested by Fig. 1, transverse edge conditions must be specified, and the radial eigenfunctions are no longer the infinite-plane oscillator modes.

   For example, in the infinite transverse trap the corrected levels can be organized by $N = 0, 1, 2, \ldots$, with

$$\mu_N^2 = m^2 + 2\alpha(N+1), \qquad d_N = N + 1.$$

   Up to separately defined self-energies, the Dirichlet plate-interaction energy would then take the standard form

$$E_{\mathrm{int}}(L) = -\frac{1}{2\pi} \sum_{N=0}^{\infty} (N+1)\mu_N \sum_{p=1}^{\infty} \frac{K_1(2pL\mu_N)}{p},$$

   with no factor A. Alternatively, the authors must formulate a finite transverse domain or a physical density of independent traps. These are different systems and cannot be interchanged by a degeneracy prescription.

3. **The split into a Landau-like term and an additional term is not a physical decomposition.**

   Equation (27) simply separates the $l = 0$ oscillator modes from all $l \neq 0$ modes. Calling the former a Landau sector does not make the latter a new correction generated by the mass profile. In a uniform magnetic field, the symmetric-gauge spectrum contains the combination $2n + |l| - l + 1$, and its infinite degeneracy is a consequence of magnetic translation symmetry. The radial oscillator instead depends symmetrically on $|l|$ and has finite shell degeneracy. Agreement of the $l = 0$ subsequence with the functional form of a Landau level is therefore not an equivalence of spectral problems.

   The issue is especially transparent in Cartesian oscillator quantum numbers, where there is no natural counterpart of the proposed $E_{\mathrm{L}} + E_{\mathrm{c}}$ separation. The ratio plotted in Fig. 3 consequently has no basis-independent physical significance. Its approach to unity at large coupling merely reflects dominance by the lowest angular-momentum states with the smallest gaps. The factor-of-two discussion for real versus complex scalars does not supply the missing magnetic degeneracy. The claims of a distinct additional Casimir contribution and of Landau-sector dominance should be removed or reformulated after the correct spectrum and observable have been established.

4. **Equations (27), (29), and (36) use mutually inconsistent mode ranges.**

   Splitting the original sum gives

   $$\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} = \sum_{n=0}^{\infty}\Big|_{l=0} + 2\sum_{n=0}^{\infty}\sum_{l=1}^{\infty}.$$

   The second terms in equations (27) and (29) instead begin at $n = 1$, thereby deleting every mode with $n = 0$ and $l \geq 1$. On the other hand, the factor $1/[\sinh(w)(e^{w} - 1)]$ in equation (36) is the result obtained when the n-sum begins at zero. If the displayed lower limit $n = 1$ were used, an extra factor e^{-2w} would be present. Thus the zeta function that is written and the heat kernel that is evaluated do not describe the same mode set, even before correcting the spectrum.

5. **The subtraction should be defined as a plate interaction, not by a rule that vacuum energy cannot grow with α.**

   The physical motivation after equations (32) and (36) is not a valid general renormalization condition. A vacuum effective action in an external background can grow with the background strength; bulk and boundary local terms are then handled by the corresponding counterterms. In the Poisson decomposition, the L-independent term represents a self-energy term, whereas the term proportional to L is the bulk contribution. The manuscript describes the former as bulk and removes both partly because of their growth with α.

   There is a clean result in the last Poisson term because it is the two-boundary interaction contribution. The authors should define it by an explicit spectral difference, such as subtraction of the infinite-separation configuration or a piston construction, and state what is held fixed in that limit. This would also distinguish a Casimir interaction energy from the full renormalized vacuum energy in a spatially varying background. The large-α decay of the interaction then follows from the growing spectral gap, not from a general prohibition on vacuum energies that grow with a classical parameter.

The spectral-zeta discussion also needs correction. The sum is over the three-dimensional spatial operator, not a four-dimensional elliptic operator. For the full trapped spectrum, the number of eigenvalues below a large cutoff scales as the cutoff to the power 5/2; the unseparated zeta series therefore initially converges for $\operatorname{Re} s > 5/2$, not for $\operatorname{Re} s > 2$ as stated below equation (28).

6. **The strong-coupling asymptotic formula in equation (39) is quantitatively wrong.**

   Even accepting the manuscript's preceding definitions, its middle line can be written with $M_f = \sqrt{m_0^2 + f\alpha_0}$ as
$$\mathcal{I}_f \simeq \frac{2f\alpha_0 M_f}{j} K_1(2jM_f).$$
   Using $K_1(x) \sim \sqrt{\pi/(2x)}e^{-x}$ gives
$$\mathcal{I}_f \sim \sqrt{\pi}\, f\alpha_0 M_f^{1/2} j^{-3/2} e^{-2jM_f}.$$
   If $f\alpha_0 \gg m_0^2$, this becomes
$$\mathcal{I}_f \sim \sqrt{\pi}\, f^{5/4}\alpha_0^{5/4} j^{-3/2} e^{-2j\sqrt{f\alpha_0}}.$$
   The last line of equation (39) instead has an $\alpha_0^{3/2} j^{-1}$ prefactor and $e^{-2jf\alpha_0}$. It omits the inverse square root of the Bessel argument and replaces $\sqrt{\alpha_0}$ in the exponent by α_0. The stated condition $\alpha_0 \gg m_0$ is likewise not the relevant one; one needs both a large Bessel argument and, for the simplified massless scaling, $f\alpha_0 \gg m_0^2$. Exponential suppression survives qualitatively, but its scale and prefactor differ parametrically from the paper's result.

7. **The weak-coupling expansion in equations (40)–(43) loses two powers of the integration variable.**

   After correcting the notational error in equation (37), define $w = \alpha_0\tau^2$. The additional integral used in the manuscript is
$$\mathcal{I}_{\mathrm{c}} = 2\alpha_0 \int_0^\infty \frac{\mathrm{d}\tau\, \tau^{-3} e^{-m_0^2\tau^2 - j^2/\tau^2}}{\sinh(w)(e^w - 1)}.$$
   Since
$$\frac{1}{\sinh(w)(e^w-1)} = \frac{1}{w^2} - \frac{1}{2w} - \frac{1}{12} + O(w),$$
   the leading term is
$$\mathcal{I}_{\mathrm{c}} \sim \frac{2}{\alpha_0}\int_0^\infty \mathrm{d}\tau\, \tau^{-7} e^{-m_0^2\tau^2 - j^2/\tau^2} = \frac{2}{\alpha_0}\frac{m_0^3}{j^3}K_3(2jm_0),$$
   not the K_2 expression in equation (43). In the massless limit the leading term is $2/(\alpha_0 j^6)$, rather than $1/(\alpha_0 j^4)$.

   The next term is also important. Writing $J_3 = (m_0^3/j^3)K_3(2jm_0)$ and $J_2 = (m_0^2/j^2)K_2(2jm_0)$, one obtains
$$\mathcal{I}_{\mathrm{c}} = \frac{2}{\alpha_0}J_3 - J_2 + O(\alpha_0), \qquad \mathcal{I}_{\mathrm{L}} = J_2 + O(\alpha_0^2).$$
   Thus the finite K_2 terms cancel in the sum used in equation (38); the statement that the next-to-leading total contribution reproduces the standard parallel-plate energy does not follow. Equations (42) and (43), Figs. 2 and 3, and all associated analytical checks must be redone.

8. **The interpretation of the $\alpha \to 0$ limit does not resolve the inconsistency.**

   The oscillator eigenfunctions exist for every $\alpha > 0$, and the continuum limit is approached through a changing spectral density as $\alpha \to 0$. The fact that one cannot set $\alpha = 0$ directly in a polynomial quantization condition does not by itself make a divergent observable physical. For the infinite transverse trap, the confinement length grows and an increasing number of low-lying transverse states contribute. For a fixed finite plate area, by contrast, the radial domain and boundary conditions remain fixed and the free-field result should be recovered with the correct density of states. The divergence attributed to the "additional sector" is therefore a warning that the geometry, order of limits, and state-counting prescription have not been held fixed consistently. These choices must be made before any weak-coupling statement can be assigned physical meaning.

9. **Novelty and phenomenological significance need to be reassessed after the calculation is corrected.**

   Equation (2) is most directly a nondynamical, spatially varying mass-squared term, or equivalently a quadratic external scalar potential. It is not the position-dependent kinetic mass operator studied in the semiconductor references cited in the Introduction, so those analogies require sharper qualification. The cited Klein-Gordon oscillator and position-dependent mass literature is closer to the actual spectral problem, but the manuscript does not explain precisely what new spectral or field-theoretic ingredient is present. The cited magnetic-field Casimir papers already contain the calculation identified here as E_{L}, while E_{c} is merely the remaining angular-momentum content of the oscillator and is presently counted incorrectly.

   After reformulating the observable, the authors should compare directly with vacuum determinants and Casimir interactions in harmonically confining or spatially inhomogeneous scalar backgrounds, explain what physical geometry could realize the proposed profile, and identify a result not already implicit in the standard oscillator heat kernel. In the current version, the claimed new Landau connection and new additional contribution do not establish sufficient novelty or significance.

Further comments

- Equation (18) leaves the normalization N unspecified. The Klein-Gordon normalization and radial orthogonality should be shown before canonical commutators and the one-mode-one-oscillator Hamiltonian are used. Doing so would also make the absence of an area-dependent degeneracy clear.
- After the change of variable $\tau \to L\tau$, both denominator factors in equation (37) must contain α_0, not the dimensionful α. As printed, that equation is dimensionally inconsistent with its exponent and with equation (35).
- The weak-coupling formulas should be written as asymptotic expansions with a stated domain and controlled remainder. Special care is required for the massless case and for interchanging the expansion with the infinite mode sum.
- The statement following equation (15) that the harmonic potential is a "relativistic correction" because its displayed expression contains c depends on how the dimensionful parameter α is

restored. The physical quantity held fixed in the nonrelativistic limit and the regime $\epsilon \ll mc^2$ should be stated explicitly.

- If the paper is intended to address a Casimir force, the pressure must be obtained by differentiating at fixed physical m and α. Because $m_0 = mL$ and $\alpha_0 = \alpha L^2$, all of their implicit L-dependence contributes. For the radially inhomogeneous background, the authors should also clarify whether a local, integrated, or averaged force is meant.
- The numerical work behind Figs. 2 and 3 needs a reproducible account of the quadrature, truncation of the image sum, convergence tests, and error estimates. Agreement between plots and formulas derived from the same incorrect spectrum is not an independent validation.
- The exposition would benefit from a careful language edit. Examples include "commom," "funtion," and repeated uses of "an harmonic." The preamble also loads `graphicx` and `hyperref` twice. These are minor matters compared with the physical and mathematical issues above.

Conclusion

The action in equation (1) can support a useful exactly solvable model, but the manuscript does not presently solve or count that model correctly. The factor-of-two spectral error, spurious area degeneracy, inconsistent geometry, nonphysical Landau decomposition, and incorrect weak- and strong-coupling limits jointly invalidate the reported energies and figures. A publishable version would require a new calculation built from a clearly defined geometry and interaction observable, followed by a renewed assessment of novelty and physical relevance.