Referee Report

Defects extremal surfaces and interface CFTs

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Recommendation: Major revision

Summary and overall assessment

The manuscript proposes a defect extremal surface (DES) prescription for a thin-wall holographic interface CFT formed by two asymptotically AdS_3 regions with generally unequal radii and central charges. Conformal matter is placed on the common end-of-the-world brane, the bulk is partially reduced near the brane in the large-tension regime, and a two-bath version of the DES functional is proposed. The authors then analyze finite intervals in a static geometry and semi-infinite radiation regions in a time-dependent geometry. In each setting they discuss no-island, ordinary-island, and double-crossing induced-island candidate saddles. The stated conclusions are that the higher-dimensional DES calculation reproduces the effective island calculation and that the induced saddle never changes the Page curve.

The subject is timely and potentially suitable for a journal such as JHEP or PRD. A controlled DES prescription for permeable interfaces could provide a useful common language for unequal-central-charge baths, refracting extremal surfaces, brane matter, and replica saddles. Several parts of the calculation are also internally consistent. In particular, the junction conditions in equations (2.20)–(2.23), the cancellation leading from equations (3.5) and (3.7) to the effective term in equation (3.8), the no-island entropy, and the conventional static island extremum have the expected form. Equation (5.16) follows algebraically by equating the candidate entropies in equations (5.7) and (5.10).

However, the present version does not yet establish its central correctness and novelty claims. There is a concrete error in both time-dependent bulk functionals: equations (5.11) and (5.15) contain $\sinh\rho$ where the brane embedding requires $\tanh\rho$. As printed, these formulas do not reproduce equations (5.10) and (5.14). Independently, the point asserted to extremize the dynamical induced-island entropy is not stationary for any finite c_{II}. The OPE/BOE channel used for that branch is also extrapolated outside its domain in Figure 8. In the static problem, the manuscript finds candidate extrema but does not establish their admissible domains, local stability, or global dominance. At a more conceptual level, equation (3.15) is postulated by analogy with equation (2.15), while the defect theory and the factorization across a permeable interface are insufficiently specified. Finally, much of the geometry, partial reduction, induced-island topology, and Page-curve analysis is attributed in the manuscript itself to earlier work; the distinct advance of the present DES reformulation is not sharply demonstrated.

I therefore recommend major revision. The underlying idea may be publishable, but the dynamical formulas and saddle analysis must be corrected, the generalized prescription must be derived and scoped, and the authors must identify a genuinely new controlled result.

Correctness and logical structure

The paper has a sensible broad progression: Section 2 reviews the one-sided DES construction and the thin-wall ICFT geometry; Section 3 adds brane matter, performs a frozen-brane partial reduction, and states the proposed generalized functional; Sections 4 and 5 test candidate saddles in static and dynamical settings. The difficulty is that several steps connecting these layers are assumed rather than derived. In particular, density-matrix factorization is inferred from a choice of large-c twist-field channel, the brane matter entropy is assigned the bath central charges without a microscopic definition, and extrema found within approximated channels are treated as complete phases without enforcing channel validity and the final minimization.

The most serious technical problems occur in the time-dependent section, where the displayed bulk distance is inconsistent with the stated embedding and the induced saddle is not an extremum of the displayed field-theory functional. These are not matters of presentation: they directly affect the advertised bulk/effective agreement and the claimed phase structure. The static results are more plausible, but their existence and dominance conditions remain incomplete.

Novelty and significance

The manuscript cites Deng, Chu, and Zhou for the original DES functional and its Page-curve application; Anous, Meineri, Pelliconi, and Sonner for the two-AdS interface geometry, finite and semi-infinite interval entropies, induced islands, and multiple-crossing surfaces; and Afrasiar *et al.* for the lower-dimensional interface reduction, replica saddles, and time-dependent island physics. Section 3.2 also states that equations (3.5)–(3.9) follow from the latter analysis after setting brane fluctuations to zero. The proposed equation (3.15) has the same formal structure as equation (2.15), with Γ replaced by $\Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{II}}$.

Accordingly, the potentially new content is narrower than the abstract and conclusion suggest: it appears to be the inclusion of a particular brane conformal-matter entropy in the DES description, together with the claim that the induced saddle is universally subdominant. Both could be interesting, but neither is yet established at journal standard. The former requires a definition and replica derivation; the latter requires a corrected, channel-consistent global phase analysis. A revision that supplies these ingredients, or another genuinely new controlled test such as finite-tension corrections or a nontrivial specified defect CFT, would substantially strengthen the work.

Major comments

1. **The time-dependent bulk geodesic terms in equations (5.11) and (5.15) are incorrect as printed.**

   The manuscript states that the Euclidean brane obeys

$$\tau = -z \sinh \rho.$$

   With the brane coordinate used in the same calculation, $z = y \operatorname{sech} \rho$, so a point on the brane has

$$(\tau, z) = (-y \tanh \rho,\ y \operatorname{sech} \rho).$$

The boundary and brane points entering the chordal distance are, respectively,

$$P = (x_1, \tau_1, \epsilon), \qquad Q = (x_b, -y \tanh \rho, y \operatorname{sech} \rho).$$

The numerator of their chordal distance is therefore

$$(x_1 - x_b)^2 + \tau_1^2 + y^2 + 2\tau_1 y \tanh \rho,$$

using $\tanh^2 \rho + \operatorname{sech}^2 \rho = 1$. Equations (5.11) and (5.15) instead contain $2\tau_1 y \sinh \rho$.

This difference is substantive. At the ordinary-island point $x_b = x_1$, $y = \tau_1$, the printed first logarithm in equation (5.11) contains

$$\log \left[\frac{2\tau_1(1 + \sinh \rho)}{\epsilon \operatorname{sech} \rho} \right],$$

which has the wrong large-ρ behavior and cannot yield equation (5.10). With $\tanh \rho$, one instead uses

$$\frac{1 + \tanh \rho}{\operatorname{sech} \rho} = e^{\rho},$$

and the expected matching follows. The same correction is needed in equation (5.15). The authors should rederive both geodesic functionals from the embedding, correct the formulas, and then repeat the extremization and all subsequent comparisons. Until this is done, the claimed exact bulk/effective agreement in Section 5 is false for the equations actually displayed.

2. **The dynamical induced-island point is not an extremum of equation (5.13).**

Retaining only the dependence on (x_b, y) in equation (5.13), define

$$D = (x_1 - x_b)^2 + (\tau_1 + y)^2.$$

The stationarity equations implied by the printed entropy are

$$3\,\partial_{x_b} S_{\text{gen}} = 2c_{\text{I}} \frac{x_b - x_1}{D} + \frac{c_{\text{II}}}{x_b}, \qquad 3\,\partial_y S_{\text{gen}} = 2c_{\text{I}} \frac{\tau_1 + y}{D} - \frac{c_{\text{I}} + c_{\text{II}}}{y}.$$

At the point asserted immediately below equation (5.13), $(x_b, y) = (x_1, \tau_1)$, these derivatives are

$$\frac{c_{\text{II}}}{x_1} \quad \text{and} \quad -\frac{c_{\text{II}}}{\tau_1},$$

respectively. They do not vanish unless $c_{\text{II}} = 0$. Thus the statement that this point extremizes S_{gen} for $c_{\text{I}} \gg c_{\text{II}}$ is, at most, a leading-order approximation in $q = c_{\text{II}}/c_{\text{I}}$, not an exact result.

For example, a direct perturbative solution begins as

$$x_b = x_1 - 2q\,\frac{\tau_1^2}{x_1} + O(q^2), \qquad y = \tau_1\big(1 + 2q + O(q^2)\big),$$

subject to the corresponding geometric domain. It is possible that substituting the shifted saddle changes the entropy only at $O(c_{\text{II}}^2/c_{\text{I}})$, so that equation (5.14) remains valid to a stated order. That point must be demonstrated, however, rather than called an exact extremum. The authors should solve the stationarity equations, test the Hessian and admissibility, state the retained order in q, and recompute equation (5.14), equation (5.17), and Figure 8 consistently. The numerical choice $c_{\text{II}}/c_{\text{I}} = 2/15$ in Figure 8 also warrants an estimate of the neglected correction.

3. **The OPE/BOE domain and the dynamical Page-curve argument must be made self-consistent.**

   For the brane endpoints $J, K = (\pm x_b, y)$, the relevant half-plane cross ratio is

$$\eta = \frac{x_b^2}{y^2}.$$

   At the approximate point used in the manuscript, the coordinate transformations give

$$\eta = \frac{x_1^2}{\tau_1^2} = \frac{\cosh^2 T}{\sinh^2 X}.$$

   Consequently, the $\eta > 1$ branch of equation (3.17), which is used to obtain equations (5.10) and (5.14), is valid only on the appropriate side of

$$T = \cosh^{-1}(\sinh X),$$

   namely the time printed in equation (5.17). Figure 8 nevertheless extrapolates the induced-island formula to earlier times, where it is in the wrong channel. The footnote below equation (5.11) cites the static calculation to dismiss $\eta < 1$, but the dynamical functional, endpoint geometry, and stationarity equations are different; this is not a derivation.

   The authors should extremize the dynamical $\eta < 1$ branch directly and determine whether it has a physical stationary point. They should then compare every admissible branch with both the no-island and ordinary-island entropies. Equation (5.17) alone only compares the two island candidates, and Figure 8 treats one of them outside its derivation domain. A proof of universal induced-saddle subdominance must also impose the junction constraints, the large-tension and $c_{\mathrm{I}} \gg c_{\mathrm{II}}$ assumptions, the reality condition for $\cosh^{-1}(\sinh X)$, and the channel inequalities. If the intended statement is restricted to the parameter choice in Figure 8 or to a controlled asymptotic regime, the abstract and Section 6 should say so.

4. **The static calculation identifies candidate extrema, not a completed three-phase entropy.**

   In Section 4.2 the ordinary-island point $y_1 = x_1$, $y_2 = x_2$ lies in the assumed $\eta > 1$ channel only if

$$\frac{(x_2 - x_1)^2}{4x_1x_2} > 1 \quad \Longleftrightarrow \quad \frac{x_2}{x_1} > 3 + 2\sqrt{2}.$$

   This explicit existence condition is not stated. For the induced candidate in Section 4.3, Appendix B obtains the cubic equation (B.3) and then replaces it by the linear equation (B.4) under $x_2 \gg x_1$ and $c_{\mathrm{I}} \gg c_{\mathrm{II}}$. A positive root alone is insufficient: one must enforce $y_1 < y_2$, the chosen OPE/BOE channel, and local minimality. At the approximate root

$$k_s = \frac{c_{\mathrm{I}} + c_{\mathrm{II}}}{c_{\mathrm{I}} - c_{\mathrm{II}}},$$

   the $\eta > 1$ condition requires

$$\frac{x_2}{x_1} > (3 + 2\sqrt{2})\, k_s^2.$$

   The revision should analyze all relevant real positive roots of equation (B.3), evaluate their Hessians, enforce their channel and ordering conditions, and compare equations (4.4), (4.9), and

(4.15) globally. Figure 10 is useful intuition but is not a substitute for this analysis, and its parameter values should in any case be specified. A phase diagram or analytic transition curves over a controlled parameter slice would establish whether the induced candidate is ever dominant and would provide a genuinely new result.

5. **The generalized DES prescription and the subsystem being computed require a replica-level derivation.**

   Equation (3.15) is formally the original DES expression in equation (2.15) with $\Gamma = \Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{II}}$. The paper should derive, rather than postulate, the new variational problem. This should include the homology constraint, the matching or refraction conditions for a surface crossing a permeable brane, the treatment of multiple intersections, and the joint variation of the intersection points and defect entropy. It should also explain why the S_{defect} term is neither omitted nor double-counted relative to S_{eff} in equations (3.12)–(3.14).

   There is also a concrete mismatch between the general formula and the examples. Equation (3.11), and the second line of equation (3.13), introduce the induced case when A lies entirely in bath I, so no A^{II} term appears. Sections 4 and 5 instead define A with components in both baths. Their $\mathrm{CFT}^{\mathrm{II}}$ correlators in equations (4.11)–(4.13) and (5.12)–(5.13) contain the entropy of A^{II} in addition to that of the induced island. The general formula for these examples should therefore contain the corresponding $S_{\mathrm{eff}}(A^{\mathrm{II}})$ contribution. The authors should define the subsystem and replica connectivity consistently and clarify whether the mixed topology studied in Figures 4(c) and 7 is the same induced saddle discussed around equation (3.11).

   Finally, equations (3.10) and (3.11) assert a tensor-product form for reduced density matrices of two interface-coupled CFTs. Large-c factorization of a twist correlator in a selected channel does not by itself prove such Hilbert-space factorization. The replica derivation should state precisely what factorizes and how the interface gluing conditions enter.

6. **The brane conformal matter and the claimed universality of equation (3.17) are underspecified.**

   Equation (3.1) permits an arbitrary conformal Lagrangian $\mathcal{L}_\Sigma$, but equation (3.17) and Appendix A treat the brane entropy as the sum of two vacuum half-plane CFT contributions with central charges exactly c_{I} and c_{II}. The derivation also assumes identity-block dominance, unit OPE/BOE normalizations, and no additional interface-localized degrees of freedom. These assumptions are restrictive for an ICFT, where transmission data, boundary or interface entropy, and nontrivial BOE coefficients can affect the entropy.

   The authors should specify the brane Hilbert space, the relation between its central charges and those of the baths, the gluing or transparency conditions, and the state whose stress tensor merely renormalizes the tension in equation (3.2). They should also explain why the entropy is additive in precisely the form of equation (3.17). With general OPE/BOE coefficients, the transition need not occur at $\eta_c = 1$ and constant interface terms can shift both phase boundaries and the Page time. Either these data should be included or every main claim should be explicitly restricted to the special identity-block model used in Appendix A.

7. **The approximation scheme, the status of the lower-dimensional gravity, and the novelty claims must be sharpened.**

   The agreements use several assumptions simultaneously: large central charge and vacuum-block factorization, large brane tension, frozen brane fluctuations, no extra interface degrees of freedom, and, for the induced saddles, strong endpoint and central-charge hierarchies. Equations (4.14)–(4.16) use an asymptotic solution of equation (B.3), while equations (5.13)–(5.15) use a zeroth-order saddle in $c_{\rm II}/c_{\rm I}$. The repeated description of these matches as *exact* is therefore misleading. The authors should state a common expansion scheme, give error estimates, and distinguish identities within the approximated functional from controlled statements about the full model. The large-tension limit should also be described as approaching the upper bound in equation (2.23), rather than as an unbounded limit.

Moreover, equation (3.8) is a two-dimensional Einstein-Hilbert term and is topological. Brane fluctuations, which generate the JT-like dilaton sector in the cited reduction, have explicitly been frozen. The paper should explain what gravitational dynamics remain, what is being varied in the lower-dimensional theory, and why the island extremization is justified in this truncated description. This clarification is essential to the stated advantage of obtaining the result without an explicit JT action.

Finally, the Introduction and Section 6 should compare the results formula by formula with the cited work of Anous *et al.* and Afrasiar *et al.*: which parts of equations (3.12)–(3.17), the static entropies, and the dynamical entropies are genuinely new, and which are consistency checks or reinterpretations? A controlled new calculation – for example finite-tension corrections, a nontrivial specified defect theory, a full phase diagram, or a treatment with brane fluctuations – would make the contribution much clearer.

Minor comments

1. Equation (2.5) has $2z_iz_i$ in the denominator. The embedding-space distance requires $2z_iz_j$.

2. Equation (4.2) writes the scalar twist two-point function as $|z_2 - z_1|^{-2h_n}$. Since the text defines $h_n = \bar{h}_n$, the full correlator scales as $|z_2 - z_1|^{-4h_n}$. Equation (4.1) and Appendix A already use the latter normalization.

3. The rescaling $\rho_k^0/L_k \to \rho_k^0$ should be declared before formulas in which both dimensional and dimensionless versions can otherwise be confused. The notation for the static coordinates y_1, y_2 and the dynamical pair $(\pm x_b, y)$ should likewise be kept distinct when the cross ratio is discussed.

4. Figure 8 should mark the OPE/BOE admissibility boundary as well as the candidate entropy crossings. Its caption should state which junction, hierarchy, and large-tension conditions the displayed parameters satisfy. Figure 10 should list the parameters used and identify all physical roots of equation (B.3).

5. Results reproduced from Anous *et al.* and Afrasiar *et al.* should be cited at the corresponding formulas in Sections 4 and 5, not only in the introductory review. This would also help the reader identify the new DES-dependent contribution.

6. The bulk paragraph of Section 4.1 contains a malformed cross-reference, as does the end of Section 5.3. These should be corrected throughout, together with typographical errors such as *entanaglement* and subject-verb disagreements.

7. The conventional title would be *Defect Extremal Surfaces and Interface CFTs*. The present title should at least use the plural *Defect* consistently with the terminology adopted in the paper.

8. The conclusion that multiple-crossing saddles are always longer should state the assumptions under which the cited result applies, especially after conformal defect matter has been added to the brane.

Recommendation

I recommend **major revision**. The manuscript addresses an interesting problem and contains a potentially useful organizing proposal, but the central time-dependent bulk formulas are wrong as printed, the induced dynamical saddle is not stationary, and the physical channel and global phase analysis are incomplete. In addition, the generalized DES functional and defect matter model need a derivation precise enough to establish that the construction goes beyond a repackaging of known ICFT island results. A revision that corrects equations (5.11) and (5.15), solves the induced saddle in its admissible channels, completes the static and dynamical minimization, and clearly isolates a new prediction could merit publication.