

Referee Report

“Magnetic corrections to the fermionic Casimir effect with Lorentz symmetry violation and a compactified extra dimension”

arXiv:2607.15212

Summary

The manuscript studies the vacuum interaction of a charged Dirac field on $\mathbb{R}^{3,1} \times S^1$ between two parallel MIT-bag plates. The setup combines a constant magnetic field normal to the plates, a CPT-even aether coupling controlled by λ , and a quasiperiodic phase β around the compact dimension. Four representative orientations of the aether vector are considered. The timelike case is written as a Landau-level sum, converted to a proper-time representation, and Poisson-resummed over the plate modes. Under $q \ll a$, the calculation retains the lowest Kaluza–Klein (KK) mode and the first image and evaluates the remaining integral by a saddle. For $\beta = 0$, the four-dimensional zero mode is separated from the first KK correction. The spacelike cases are then generated by substitutions in the timelike expressions, and Section IV differentiates the resulting energies to obtain pressures.

The combination of magnetic, aether, and compactification effects is a legitimate subject. The manuscript also has a useful organizing idea: in the small- q/a regime, nonzero KK modes can often be interpreted as heavy four-dimensional modes and treated asymptotically. Unfortunately, the central spectral input does not describe the massive MIT-bag problem stated in the title and abstract. Several additional normalization, compact-phase, and anisotropic-Landau errors then affect even the simplified calculation. These are structural problems rather than local slips.

Recommendation

I recommend **rejection in the present form**, with encouragement to resubmit a substantially rederived paper. The decisive issue is that Eqs. (II.5)–(II.13) impose the massless half-integer plate spectrum while the paper claims results for a massive fermion, including both $am \ll 1$ and $am \gg 1$. Correcting this changes the mode sum, invalidates the Poisson resummation used in Section III, and requires recalculation of essentially all reported massive energies and pressures. A viable revision could instead be restricted consistently to $m = 0$, but it would still have to correct the five-dimensional Clifford convention, KK-mode selection, transverse aether spectrum, weak-field domain, and normalization discussed below.

Major comments

1. The plate spectrum is the massless MIT spectrum, not the massive one.

The manuscript states after Eq. (II.5) that $k_n = (n - \frac{1}{2})\pi$ and then uses $(k_n/a)^2$ in Eqs. (II.5)–(II.13). For a massive fermion satisfying the stated MIT condition (II.4), the dimensionless roots $x_n = k_{3,n}a$ instead obey, up to the equivalent sign convention for the outward normals,

$$ma \frac{\sin x_n}{x_n} + \cos x_n = 0, \quad \text{or equivalently} \quad ma \sin x_n + x_n \cos x_n = 0.$$

Half-integers follow only when $m = 0$. Moreover, when the aether vector is normal to the plates, the boundary equation itself changes; with the conventions used in the manuscript it takes the form

$$ma \sin x_n + (1 - \lambda)x_n \cos x_n = 0.$$

These transcendental conditions are not an optional refinement. They are derived in the manuscript's own Refs. [18] and [31], which are the closest predecessors for the compact massive problem.

The consequence is immediate. The half-integer Poisson formulas (III.7)–(III.8) cannot be applied to the massive roots, so the chain (III.9)–(III.22), every substitution-generated result (III.23)–(III.32), and all pressures based on them do not solve the stated massive MIT problem. The discrepancy is already visible perturbatively. If $x_n^{(0)} = (n - \frac{1}{2})\pi$, the exact root equation gives $x_n = x_n^{(0)} + ma/x_n^{(0)} + O((ma)^2)$. Thus the normal eigenvalue squared contains a term $2m/a$ before the explicit m^2 term is considered. The standard small-mass interaction energy consequently contains a correction linear in m (with the usual conventions, $+m/(24a^2)$ per unit plate area), whereas Eq. (III.19) starts at $m^2/(48a)$. The $am \gg 1$ formulas are no better justified: the roots move toward integer rather than half-integer multiples, and the bag reflection factor changes the asymptotic prefactor.

The paper must derive the magnetic and Lorentz-violating mode conditions from Eq. (II.4) and evaluate the resulting root sum by an argument-principle or appropriate Abel–Plana method. At minimum, corrected expressions should be checked against Refs. [18] and [31] when $B \rightarrow 0$. Alternatively, all massive, light-mass, and heavy-mass claims should be removed and the scope restricted explicitly to $m = 0$.

2. The five-dimensional Clifford algebra is inconsistent with a spatial compact dimension.

Below Eq. (II.1), the fifth matrix is identified with the usual four-dimensional chirality matrix, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. This matrix squares to $+1$. In the stated mostly-minus convention, however, a fifth *spatial* gamma matrix must satisfy $(\Gamma^4)^2 = -1$ as part of $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$ with $\eta^{ab} = \text{diag}(+, -, -, -, -)$. One may take, for example, $\Gamma^4 = i\gamma_{\text{ch}}^5$, up to an overall sign. With the matrix printed in the manuscript, squaring the Dirac operator gives the opposite sign for the KK term from the $+k_\ell^2$ assumed in Eqs. (II.5)–(II.13).

The five-dimensional metric, Clifford representation, adjoint, and Hermiticity conventions should be stated together and used to rederive the dispersion relations. This is especially important for the u^a component along the compact direction, whose advertised effect is entirely tied to the sign and normalization of this matrix.

3. The magnetic coupling and the transverse aether spectrum need a gauge-covariant derivation.

Equations (II.1)–(II.2) contain only partial derivatives. The magnetic field is introduced later by asserting Landau spectra (II.10)–(II.13), without writing a vector potential or specifying whether the aether derivative is also covariantized. Gauge invariance requires a definite replacement $\partial_a \rightarrow D_a = \partial_a + ieA_a$ in both kinetic terms. The squared Dirac operator should then be displayed, including its spin coupling, rather than inferred by analogy.

This omission matters most for $u = (0, 1/\sqrt{2}, 1/\sqrt{2}, 0, 0)$. A fixed transverse aether vector rescales the momentum parallel to u by $1 - \lambda$ and leaves the orthogonal momentum unchanged. In

the original x^1, x^2 coordinates, the resulting quadratic form contains a $k_1 k_2$ cross term, absent from Eq. (II.7). After rotating to parallel and perpendicular axes, the exact Landau spacing is controlled by the determinant of the transverse velocity matrix and is

$$(2j + 1 \pm 1)(1 - \lambda)|eB|$$

for $1 - \lambda > 0$, not exactly $(2j + 1 \pm 1)(1 - \lambda/2)^2 eB$ as in Eq. (II.12). These agree only through first order in λ . The manuscript later replaces $(1 - \lambda/2)^2$ by $1 - \lambda$ while retaining nonlinear λ dependence elsewhere. It must either give the exact rotated derivation or declare and enforce a consistent truncation order in λ .

All Landau degeneracies and logarithms should also use $|eB|$, unless the paper explicitly fixes $eB > 0$.

4. The zero-field and Landau normalizations differ by a factor of two.

Equations (II.5)–(II.8) use $-CL^2/(2\pi^2)$ and then set the spin factor to $C = 2$. For a Dirac field with the transverse momenta integrated over the full plane, canonical spin counting instead gives

$$-2L^2 \int \frac{d^2 k}{(2\pi)^2} \sum_n E_n = -\frac{L^2}{2\pi^2} \int d^2 k \sum_n E_n.$$

Thus the prefactor printed in (II.5)–(II.8), after inserting $C = 2$, is twice the canonical result. Independently, taking the continuum $B \rightarrow 0$ limit of the combined spin sums in Eqs. (II.10)–(II.13) produces $-L^2/(2\pi^2)$, not $-L^2/\pi^2$. The integration domains and degeneracy conventions should be stated, and the zero-field expressions made consistent with the magnetic ones.

There is a related derivational gap in the timelike case. Equation (II.2) gives a frequency proportional to $(1 + \lambda)^{-1}$, while the final vacuum energy is independent of λ . Such a cancellation can occur because the canonical Hamiltonian carries a compensating factor $1 + \lambda$, as in Ref. [17], but the present manuscript does not derive it after adding B and the compact direction. The Hamiltonian, mode normalization, and canonical anticommutators should be shown so that this cancellation and the overall mode weight are auditable.

5. The leading KK mode is selected incorrectly for general β , and the approximation is nonuniform.

The exact sum depends on $(\ell + \beta)^2$ and is invariant under $\beta \mapsto 1 - \beta$ by reindexing ℓ . For $0 \leq \beta < 1$, the smallest KK magnitude is

$$|k|_{\min} = \frac{2\pi}{q} \delta, \quad \delta = \min(\beta, 1 - \beta).$$

The leading mode is $\ell = 0$ only for $0 < \beta < 1/2$; it is $\ell = -1$ for $1/2 < \beta < 1$, and both modes are degenerate at $\beta = 1/2$. The assertion before Eq. (III.12) that $\ell = 0$ is always the lowest nonzero-twist mode is therefore false. Equation (III.15) misses a factor of two at $\beta = 1/2$ and, for $\beta > 1/2$, uses an exponentially more suppressed mode than the true leading contribution. The published formulas consequently violate an exact symmetry of the starting sum.

Moreover, $q \ll a$ alone does not imply $a^2 k_\ell^2 \gg 1$ for every $\beta \neq 0$. The relevant condition is

$$a\sqrt{m^2 + (2\pi\delta/q)^2} \gg 1,$$

and, when the KK scale dominates, $2\pi(a/q)\delta \gg 1$. This fails in a boundary layer near $\beta = 0$ or 1. Near $\beta = 1/2$, two modes may be comparable even away from exact degeneracy. Equations (III.15), (III.23), (III.27), and (III.31), together with their pressures, need piecewise mode selection, degeneracy factors, and a quantitative error estimate. A uniform formula retaining the two nearest modes would be preferable.

6. The transverse light-mass result contains an internal algebraic error.

Let $r = 1 - \lambda$. Once the transverse Landau spacing is $r|eB|$ but the Landau degeneracy remains $|eB|L^2/(2\pi)$, the manuscript's own substitution logic implies

$$E_{\perp}(B) = \frac{1}{r}E_0(rB).$$

Consequently, applying this relation to Eq. (III.19) requires

$$E_{\perp}^0 = -\frac{1}{r} \left(\frac{7}{8} \right) \frac{\pi^2 L^2}{360 a^3} + \frac{1}{r} \frac{m^2 L^2}{48 a} + \frac{r(eB)^2 L^2 a}{12 \pi^2} \left[\ln \left(a \sqrt{r|eB|} \right) + \frac{1}{8} \right]$$

within the same approximations. Equation (III.28) omits $1/r$ on both B -independent terms. Equation (IV.10) correspondingly omits $1/r$ on its zero-field leading pressure. This is also forced by the $B \rightarrow 0$ limit of $|eB| \coth(ra|eB|/M)$ in Eqs. (III.27) and (III.29), and by the $1/r$ Jacobian of the anisotropic transverse momentum. Thus the light- and heavy-mass transverse formulas do not even approach the same zero-field theory as currently printed.

7. The $B^2 \log B$ formulas have an unstated weak-field domain.

Equation (III.19) is obtained from the weak-field expansion

$$|eB|t \coth(|eB|t) = 1 + \frac{(|eB|t)^2}{3} + \dots$$

For the plate contribution, whose proper-time scale is set by a^2 , this requires approximately $|eB|a^2 \ll 1$, in addition to $am \ll 1$. The cited Majorana calculation in Ref. [36] explicitly separates weak- and strong-field regimes. In strong field the lowest Landau contribution is linear in $|eB|$, whereas Eq. (III.19) grows as $B^2 \log B$; it therefore cannot be an all-field expression.

The missing restriction propagates to Eqs. (III.24), (III.28), (IV.3), (IV.4), (IV.7), (IV.10), and (IV.13), with appropriately rescaled inequalities for the different aether orientations. The abstract and conclusion should not call the results generally accurate without listing these domains. The paper should provide either separate strong-field results or a precise scope statement, and should compare the approximations with numerical evaluation of the unexpanded proper-time expression.

8. The zeta continuation and the Casimir subtraction are not defined carefully enough.

The Mellin identity in Eq. (III.3) is an ordinary convergent integral only for $\text{Re } s > 0$, whereas Eq. (III.1) sends s directly to $-1/2$. The calculation should introduce s in a convergence half-plane, perform the mode sums and the plate-independent subtraction there, and only then analytically continue to $s = -1/2$. This is more than a formal nicety because the removed terms carry the ultraviolet magnetic and boundary divergences.

In particular, the term proportional to $L^2 a$ in Eq. (III.9) is said not to contribute because it is a uniform energy density. A term linear in a does contribute a constant to $-L^{-2} \partial E / \partial a$. It can

be removed from the *interaction* pressure only after a stated subtraction of the no-plate bulk energy, together with the appropriate exterior stresses. In a magnetic background this also fixes the treatment of the bulk Heisenberg–Euler term and charge/field renormalization. The answer depends on whether the magnetic field and the fermion field occupy the exterior regions. The paper should follow the decomposition used in Ref. [31] into bulk topological, one-plate, and two-plate interaction pieces and state exactly which quantity is called the Casimir energy.

9. Section IV presents leading saddle derivatives as exact pressures and reverses two qualitative trends.

The issue is visible already by differentiating Eq. (III.15). With $M = \sqrt{m^2 + k_0^2}$ and $b = |eB|$, that equation has the form

$$\frac{E}{L^2} = -\frac{bM^{1/2}}{4\pi^{3/2}\sqrt{a}}e^{-2aM}\coth\left(\frac{ab}{M}\right).$$

Direct differentiation gives

$$P = -\frac{bM^{1/2}}{4\pi^{3/2}\sqrt{a}}e^{-2aM}\left[\left(2M + \frac{1}{2a}\right)\coth\left(\frac{ab}{M}\right) + \frac{b}{M}\operatorname{csch}^2\left(\frac{ab}{M}\right)\right].$$

Equation (IV.2) retains only the $2M\coth(ab/M)$ term. This is consistent as the leading saddle result when $aM \gg 1$, since the omitted pieces are relatively $O((aM)^{-1})$, but it is not the equality claimed after Eq. (IV.1). The same omission recurs in the exponentially suppressed pieces of Eqs. (IV.3)–(IV.14). They should be marked as asymptotic formulas and accompanied by their error orders.

The prose then draws conclusions opposite to the displayed expressions for $0 < \lambda \ll 1$. In the normal x^3 case, the factor $\exp[-2aM/(1-\lambda)]$ in Eqs. (IV.6) and (IV.8) produces stronger exponential suppression as λ increases; in the assumed $aM \gg 1$ regime this weakens, rather than strengthens, the force. Conversely, $\coth[(1-\lambda)x]$ in Eqs. (IV.9) and (IV.11) increases as λ increases, so the printed transverse formulas strengthen rather than weaken the magnitude. These trends must be reassessed after the spectrum is corrected, and the allowed sign and range of λ should be stated.

10. Novelty and significance require validation beyond combining known ingredients.

The bibliography shows that the principal ingredients have been treated separately: aether-like fermionic Casimir systems in Refs. [17] and [18], massive MIT plates with compact dimensions in Ref. [31], and magnetic fermionic Casimir systems in Refs. [36] and [40]. Combining all three is a reasonable incremental extension, but most spacelike results in the present paper are obtained by substitutions rather than independent spectral calculations. No numerical comparison is given between the proposed one-mode/saddle expressions and the original sums, despite the repeated claim that the formulas are accurate.

A publishable revision should demonstrate analytically and numerically the $B \rightarrow 0$, $\lambda \rightarrow 0$, and $m \rightarrow 0$ limits; recover the exact compact massive results of Refs. [18] and [31] where applicable; verify the four-dimensional magnetic limit; and quantify errors from discarding higher KK and image modes. Under the assumed $q \ll a$, the genuinely new nonzero-KK effects are exponentially small, while at $\beta = 0$ the leading term is the already known four-dimensional zero mode. The paper should therefore include a realistic parameter study showing when the new directional correction is distinguishable and why it is of interest to the intended high-energy theory community.

Minor comments

1. Equation (III.4) has the wrong Landau exponent. Equation (II.10) requires $\exp[-(2j+1\pm 1)eBt]$, whereas (III.4) prints $\exp[-2(j+1\pm 1)eBt]$. The latter does not lead to the series identity (III.6); Eq. (III.5) silently uses the corrected spectrum.
2. Section V repeatedly cites Eq. (III.15) for distinct $\beta = 0$ spacelike light- and heavy-mass results. The intended equations are among (III.24)–(III.32), with (III.19)–(III.20) supplying the compact-direction zero-mode terms. The conclusion should be rewritten after the substantive corrections rather than repaired mechanically.
3. The statement that all possible directions of Lorentz violation have been examined is too broad. The paper treats pure timelike and selected pure spatial representatives, not generic mixed time-space or mixed spatial orientations.
4. Clarify the five-dimensional geometry of the plates: whether they wrap the compact circle, what quantity L^2 measures, and why the reported pressure is normalized per ordinary two-dimensional area rather than per hypersurface volume.
5. Several presentation errors should be corrected, including “dependance,” “this manipulations,” the isolated “Therefore.” in Section III, and the claim that the discarded bulk term does not contribute to the vacuum energy. These are secondary to the required recalculation.

Conclusion

The magnetic-aether-compactification combination could support a useful paper, but the present manuscript does not yet calculate the advertised massive MIT-bag Casimir effect. The half-integer plate spectrum invalidates the main derivation, and the independent Clifford, KK, transverse, normalization, and regime errors prevent the remaining formulas from serving as a reliable massless result without further work. A complete rederivation, rather than a local revision of the displayed expressions, is necessary before the work can be assessed for publication.