Referee Report

Analytical and Numerical Study of Quartic Nonlinear Electrodynamics in Holographic Fermion Systems

Summary and recommendation

The manuscript considers a charged probe spinor in a fixed planar AdS_4 black-brane geometry. The Maxwell action for the electrostatic potential is supplemented by a quartic term advertised as the leading open-string correction. For the ansatz $A_t = \phi(z)$, the authors reduce the gauge equation to a conserved radial flux and write it as the cubic constitutive relation

$$E - \gamma z^4 E^3 = \rho, \qquad E = -\phi', \qquad \gamma = \frac{3\alpha'^2}{2\beta^2}.$$

They identify a fold of this cubic, derive a limiting coupling for a branch that reaches the horizon, and obtain a small-α' expansion of the chemical potential. The resulting potential is then inserted into a Riccati flow equation for a probe Dirac field. From momentum scans of the spectral function at a small frequency, the paper reports an increasing peak position and peak height as α' approaches the limiting value.

There is a useful elementary observation in the paper: conditional on the reduced equation (2.12), the first integral and the branch analysis in Section 3 are transparent and mostly algebraically correct. In particular, equations (3.1)–(3.18) and the weak-coupling expansion through equation (3.32) follow from that reduced equation. The presentation also separates the analytic gauge calculation from the spinor calculation clearly.

Unfortunately, the action from which the entire analysis starts has a fundamental index/sign problem. The F^4 contraction printed in equation (2.9) is not the standard Abelian DBI/open-string contraction. On the purely electric background it gives a *negative* quartic correction, whereas the DBI expansion gives a positive one. This sign is precisely what creates the fold, the limiting coupling, the increase of μ, and the numerical trends emphasized in the paper. With the standard contraction the constitutive relation is monotone and the central analytic story disappears. There are also independent foundational problems: the RN charge and the probe charge density are not embedded in a consistent action, the fermion numerics omit essentially all parameters and convergence data, a broad finite-temperature peak is not sufficient evidence for a Fermi surface or increased coherence, and the manuscript does not discuss a directly relevant prior study of fermionic spectra in Born–Infeld AdS backgrounds.

I therefore recommend **rejection in the present form**. This is not a case in which local corrections would suffice: the gauge action must first be fixed or explicitly reinterpreted, after which the background, chemical potential, and fermion spectra must be recomputed. A substantially rebuilt paper could be considered as a new submission if it establishes a consistent model, provides controlled spectral evidence, and demonstrates an advance over the existing nonlinear-electrodynamics literature.

Major comments

1. **The quartic invariant in equation (2.9) has the wrong contraction for the claimed open-string/DBI correction.**

   This can be seen without relying on any convention beyond the metric used in the manuscript. Introduce the local orthonormal electric field

$$e = \frac{z^2}{\beta} E.$$

   For a purely electric field in the mostly-plus signature of equation (2.1), the invariants appearing exactly as printed in equation (2.9) obey

$$F_{\mu\lambda} F_{\nu}{}^{\lambda} F_{\rho}{}^{\mu} F^{\nu\rho} = -2e^4, \qquad \frac{1}{4}(F_{\mu\nu}F^{\mu\nu})^2 = e^4.$$

   Consequently, equations (2.7) and (2.9) reduce locally to

$$\mathcal{L}_{\mathrm{M}} + \mathcal{L}_{\mathrm{P}} = \frac{1}{2}e^2 - \frac{3\alpha'^2}{8}e^4.$$

   The Abelian DBI expansion instead has a positive e^4 correction. In the usual tensor notation its quartic invariant is proportional to $\mathrm{tr}(F^4) - \frac{1}{4}(F^2)^2$. The first contraction displayed in equation (2.9) is the negative of $\mathrm{tr}(F^4)$ on the present background; the placement of the index on the third field strength appears to be the source of the error.

   The downstream calculation is internally consistent with the *printed* negative-quartic action: varying that action indeed gives the sign in equations (2.10) and (2.12). This does not cure the physical problem, because the negative sign is also the sole origin of the maximum of $D(E) = E - \gamma z^4 E^3$. If equation (2.9) is replaced by the standard Abelian quartic contraction while retaining the normalization used in the paper, the electrostatic first integral becomes

$$E + \frac{\alpha'^2}{2\beta^2} z^4 E^3 = \rho.$$

   This map is monotone for positive α'^2: there is no E_c, no z_c, and no critical bound of the form (3.22). Moreover, the electric field and chemical potential decrease rather than increase. In fact, the corrected weak-coupling result would begin

$$\mu = \rho - \frac{\alpha'^2\rho^3}{10\beta^2} + \frac{\alpha'^4\rho^5}{12\beta^4} + \mathcal{O}(\alpha'^6),$$

   the opposite of equation (3.29) at leading order. Thus the abstract, the main results of Section 3, all backgrounds used in Section 4, and the physical conclusions would change.

   The authors should derive their quartic action explicitly from the cited open-string effective action, including signature, index positions, overall normalization, and the reduction to the electric ansatz. If the negative quartic theory is intentional, it must instead be presented as a phenomenological nonlinear-electrodynamics model rather than as the leading string correction. In that case the manuscript also needs a stability and causality analysis, as discussed below. Either resolution requires a full recalculation rather than a textual clarification.

2. **The RN geometry and the nonlinear gauge field do not yet define a consistent bulk model.**

   Equations (2.1)–(2.4) describe a charged RN–AdS black brane with charge parameter q. Equations (2.6)–(2.15), however, introduce the only displayed bulk gauge field with an independent density ρ, change its equation by the quartic interaction, and state that this field does not backreact. These statements cannot all refer to a single $U(1)$. If A_μ supports the RN geometry, its charge is related to q, and both the metric and the potential must change when the gauge Lagrangian changes. If A_μ is a separate probe field, then a second gauge field that supports the RN metric must appear in the action. The boundary theory then has two currents, and the charge of the probe fermion under each current must be stated.

   The numerical values in Table 1 suggest a third, simpler possibility. With $\beta = \rho = 1$, its Maxwell value $T/\mu = 0.238$ is, to the displayed precision, $3/(4\pi)$, which corresponds to $q = 0$. If this inference is correct, the numerical geometry is actually AdS–Schwarzschild and the finite density is entirely in a probe gauge sector. That is a legitimate bottom-up limit if an explicit ratio of gravitational and gauge couplings suppresses the stress tensor. It is not the charged RN setup described in Section 2, and the action as written contains no parameter that controls this limit.

   The revised work must write the complete action, including the cosmological term, gravitational and gauge couplings, AdS radius, and any second $U(1)$. It must state which ensemble is used and how q, ρ, and μ are related or kept independent. If all numerical results use $q = 0$, the RN discussion should be removed or replaced by a genuine scan showing the role of q. Until this point is resolved, the purported dual finite-density system and the meaning of T/μ are ambiguous.

3. **The fermion action and the radial flow use incompatible factors of i.**

   Equations (2.16) and (2.18) use the operator $i\Gamma^{\underline{a}} e_{\underline{a}}{}^{\mu} D_\mu - m$ with a Clifford basis adapted to mostly-plus signature, for which $(\Gamma^{\underline{t}})^2 = -1$ and $(\Gamma^{\underline{z}})^2 = +1$. The subsequent UV powers and Riccati equation are instead the standard results from $(\Gamma^{\underline{a}} e_{\underline{a}}{}^{\mu} D_\mu - m)\psi = 0$. For example, with the operator as printed, the leading radial equation near the boundary contains $iz\Gamma^{\underline{z}}\partial_z - m$ and gives imaginary mass exponents rather than the real $z^{\pm m}$ behavior in equation (2.24). The mass term in equation (2.23) would likewise carry an extra factor of i.

   In a common holographic convention the Lorentzian spinor action has an overall prefactor i multiplying $\bar{\psi}(\Gamma^{\underline{a}} e_{\underline{a}}{}^{\mu} D_\mu - m)\psi$; the overall factor does not enter the equation of motion. Other conventions are possible, but equations (2.16), (2.18), (2.23), and (2.24) must all follow from one of them. The authors should correct the action and rederive the flow, including charge and momentum signs. If the numerical calculation uses $m = 0$, this particular error may be hidden in the displayed plots, which makes the omission of the numerical value of m still more consequential.

4. **Even as an intentional negative-F^4 toy model, the branch near the claimed endpoint is not shown to be a controlled or stable effective theory.**

   Conditional on equation (2.12), the algebraic manipulations in Section 3 are correct: differentiating equation (3.7) gives equation (3.13), and the fold conditions (3.14)–(3.18) follow. The physical interpretation, however, requires more than static invertibility. The same quantity

   $$\frac{\partial D}{\partial E} = 1 - 3\gamma z^4 E^2$$

   enters the quadratic response of the gauge field. Its approach to zero is a strong-coupling or characteristic-degeneracy warning for fluctuations, not merely a vertical tangent in a background curve. The manuscript provides no analysis of the kinetic matrix, hyperbolicity, ghosts, characteristic cones, or causal propagation on the chosen branch.

There is also a branch-counting error in the static discussion. Below the fold, $E - aE^3 = \rho$ with $a = \gamma z^4 > 0$ has two positive roots and one negative root, not a "unique positive solution" as stated after equation (3.19) and in the caption of Figure 2. Only the lower positive root, $0 < E < 1/\sqrt{3a}$, is continuously connected to the Maxwell solution. The revised text must distinguish uniqueness on this selected branch from the number of algebraic roots, and the numerical root-selection procedure must be specified.

This concern is most acute for the advertised fermion results. Table 1 and Figures 3–4 emphasize $\alpha' = 0.31$, while equation (3.23) gives $\alpha'_{\rm crit} \simeq 0.314$ for the inferred unit parameters. Near that endpoint the quartic term is not a small correction in the infrared, the background is highly sensitive to numerical and higher-order changes, and neglected F^6 terms need not be suppressed. Backreaction is also not quantified. Calling this a controlled leading string correction is therefore not justified even apart from the sign problem.

If the negative-quartic model is retained, the authors should derive the full quadratic fluctuation operator and establish the domain in which it is hyperbolic and ghost-free. They should display a dimensionless EFT expansion parameter over the complete radial interval and restrict quantitative claims to a region in which omitted operators and backreaction are demonstrably small. The smoothness bound is also strict: $z_c > 1$ implies $|\alpha'| < \sqrt{8/81}\,\beta/\rho$. Equality in equation (3.22) places the divergent derivative of equation (3.13) at the horizon and is not a globally smooth solution.

5. **The fermion calculation is not reproducible from Section 4.**

   None of the numerical figures or Table 1 states the complete parameter set. At minimum the calculation depends on m, q_f, q, β, ρ, and the nonzero frequency represented by "$\omega \simeq 0$." These values are not supplied. The manuscript also omits the horizon cutoff ϵ, the UV cutoff, the integration algorithm and tolerances, the method used to select the Maxwell-connected root of the cubic, the momentum grid and peak interpolation, and numerical errors. Figure 2 does not even state the value of α' or γ. The sentence at the end of Section 4.2 says that spectra at different α', q, and β are compared, but the paper presents only an α' scan.

   The missing limiting procedure is especially serious. The infalling data in equations (2.26) and (4.3) are explicitly stated only for $\omega \neq 0$, while the Fermi momentum in equation (4.8) is extracted at an unspecified frequency near zero. The flow is singular at the horizon, and setting $\xi_I = i$ at $z = 1 - \epsilon$ without a controlled near-horizon expansion can produce a result that depends on the ratio of ω to ϵ. This sensitivity is enhanced close to the constitutive fold.

   A publishable numerical section should give a parameter table and enough algorithmic detail to reproduce every plotted curve. It should demonstrate convergence under independent changes of ω, the horizon and boundary cutoffs, solver tolerances, and momentum resolution; provide uncertainties on k_F and peak observables; and state whether the chemical potentials in Table 1 are obtained from the exact cubic solution or from the truncated series (3.29). Releasing the data and a short integration script would be the clearest way to make the calculation auditable.

6. **A maximum of a broad finite-temperature momentum scan does not establish the claimed robust Fermi surface or increasing coherence.**

Equation (4.8) defines k_F as the maximum of $A(\omega, k)$ at an unspecified small but nonzero frequency. Figure 3 shows a broad peak with a finite height of order five or six at an apparent $T/\mu \simeq 0.23$. Such a peak is a useful finite-temperature crossover diagnostic, but by itself it is not a zero-frequency pole, a normal mode, or an infinitely sharp Fermi-surface singularity. This distinction matters because the classic RN–AdS fermion analysis identifies the Fermi surface from the $\omega \to 0$ pole structure, often together with the near-horizon AdS_2 exponent. If the inferred $q = 0$ setup is used here, that RN AdS_2 structure is absent altogether.

The claims that the Fermi surface is "robust" and that the excitation becomes "more coherent" therefore go well beyond the evidence shown. The peak height $A_{\max}$ is also not a spectral weight: spectral weight is associated with an integrated area or a pole residue, whereas a height can increase solely because a peak narrows or because the frequency regulator changes. No width, residue, lifetime, dispersion relation, or quasinormal-mode pole is computed.

The authors should show $A(\omega, k)$ over a two-dimensional neighborhood of the proposed Fermi point, track the dispersion and linewidth, test the $\omega \to 0$ limit, and identify poles or normalizable zero modes. At finite temperature the language should be reduced to a spectral peak unless a precise operational definition of a thermally broadened Fermi surface is given. Any statement about coherence should be based on a width, lifetime, or pole residue rather than on $A_{\max}$ alone.

7. **The numerical scan does not disentangle a change of scale from a change of the radial profile.**

   All reported spectra appear to vary α' at fixed ρ, β, and background temperature. Equations (3.29) and (3.32) then show that μ and T/μ change at the same time as the shape of $\phi(z)$. The manuscript attributes the increase of $A_{\max}$ mainly to the decrease of T/μ and the shift of k_F to the deformation of the full radial profile. No control calculation in Section 4 separates these mechanisms.

   In particular, k_F is a dimensionful momentum in the conventions used. A change in the overall chemical-potential scale can shift it even if the dimensionless spectral physics is otherwise unchanged. The primary comparison should therefore include k_F/μ, ω/μ, the peak width in units of μ, and appropriately normalized residues. More decisively, the authors should compare nonlinear and Maxwell profiles at matched μ and T/μ, or construct a Maxwell control potential with the same boundary value and isolate the residual effect of the radial shape. A complementary fixed-μ rather than fixed-ρ scan would expose the ensemble dependence. Without such controls, the two-mechanism interpretation in Sections 4.2–4.3 and the conclusions is conjectural.

8. **The novelty assessment omits direct prior work on nonlinear electrodynamics and holographic fermion spectra.**

   The Introduction cites nonlinear-electrodynamics studies of holographic superconductors, but it does not cite the directly relevant paper by J.-P. Wu, "Holographic fermionic spectrum from Born–Infeld AdS black hole," arXiv:1705.06707. That work studies relativistic and nonrelativistic probe fermion spectra, Fermi momenta, and nonlinear-gauge dependence in a Born–Infeld AdS black-hole background, including backreaction of the nonlinear gauge sector on the geometry. It is therefore much closer to the present subject than several references currently used to motivate novelty. It also reports a different trend of the relativistic Fermi momentum with its Born–Infeld parameter, making a careful comparison of sign conventions, ensemble, and backreaction particularly important.

   The current analytic additions do not by themselves establish sufficient novelty for JHEP or Physical Review D. Radial Gauss-law conservation and a first integral are generic for a homogeneous electrostatic ansatz in nonlinear electrodynamics. Likewise, dependence on even powers of α' is built into a model whose equations contain only α'^2. After fixing the action, the authors should survey the nonlinear-electrodynamics fermion literature, explain precisely what the truncated quartic calculation adds to the Born–Infeld analysis, and compare observables in common dimensionless variables and ensembles. The novelty claim in the present Introduction is not adequately supported.

9. **The fits in equations (4.9)–(4.10) do not demonstrate an independent even-power prediction.**

   The paper treats the agreement of k_F and $A_{\max}$ with quartic polynomials in α' as a substantive analytic confirmation. However, the background and spinor equations depend on α' only through $\gamma \propto \alpha'^2$. Any analytic observable is therefore a function of γ and is even in α' by construction. Since the numerical data sample only $\alpha' \geq 0$, they cannot test parity independently. The fits are best regarded as interpolations, not as evidence for a new physical law.

   There is an additional tension with the claimed critical point. A fold of the algebraic branch generically produces nonanalytic square-root behavior as the endpoint is approached, so a low-order polynomial in γ has no controlled reason to describe the near-critical region. No residuals, uncertainties, covariance of coefficients, or comparison with lower-order fits is shown, and the fitted intercepts do not exactly reproduce the rounded Maxwell entries in Table 1. More substantially, equation (4.9) gives $k_F(0.20) \simeq 0.8121$ rather than the tabulated 0.804, while equation (4.10) gives $A_{\max}(0.31) \simeq 6.40$ rather than 6.47. Figure 4 also ends at $\alpha' = 0.30$ although Table 1 includes 0.31. If these fits are retained, the authors should fit directly in γ, publish residuals and errors, justify the fit range, and avoid describing automatic evenness as a numerical discovery. The near-endpoint behavior should instead be analyzed using the actual branch structure.

Minor comments

1. The heading of Section 2 reads "Holographic Mode" and should read "Holographic Model."

2. The paper should define the dimensions of α', β, ρ, q, and q_f, state the AdS-radius convention, and identify the actual dimensionless expansion parameter. Treating numerical values such as $\alpha' = 0.31$ as meaningful requires these conventions.

3. Equation (2.6) is not a complete action: the Einstein–Hilbert and cosmological terms, the integration measure, and the relative gravitational, gauge, and fermion normalizations are needed. The boundary term and quantization choice for the spinor should also be stated when defining the Green function in equation (2.25).

4. The gamma-matrix basis in equation (2.20) does not specify $\Gamma^{\underline{y}}$. Although momentum is chosen along x, the complete four-dimensional Clifford representation should be given.

5. The quantity $E = -\phi'$ in equation (3.6) is a coordinate component. The local orthonormal electric field is $z^2 E/\beta$. Accordingly, the "dielectric functions" in equations (3.10)–(3.11) should be described as reduced radial response functions unless an invariant constitutive tensor is derived.

6. Every figure caption should state all parameters. Figure 1 includes an $\alpha' = 0.6$ curve that terminates before the horizon, while the fermion analysis is restricted to global solutions; this distinction should be made visually explicit. Figure 2 needs the values of ρ, β, and the coupling for each curve.

7. Table 1 should quote sufficient digits to distinguish exact and perturbative values of μ and should include numerical uncertainties for k_F and A_{max}. The value of the small frequency used to define the table must appear in its caption.

8. The manuscript refers in prose to Sections II–V, whereas the compiled article numbers them 2–5. These references should be generated consistently. The bibliographic entry for the 2012 open-string effective-action paper should also include complete volume, page/article number, and arXiv information.