

Referee Report

Relativistic time-commutative dynamics with κ -plane noncommutativity

Summary and recommendation

The manuscript develops a first-quantized framework intended to connect Hopf-algebra spacetime symmetries with self-adjoint quantum-mechanical observables. After reviewing quantum differential calculi, the authors distinguish the abstract Poincaré Hopf generators from a set of canonical generators acting by inner commutators on a covariant quantum-mechanical algebra. They then specialize to the time-commutative κ -plane, construct a proposed Poincaré Hopf algebra through first order in the noncommutativity length ℓ , realize the associated canonical generators, and identify the invariant constraint in equation (99). Contracting to the Galilean regime gives the canonical algebra in equations (110)–(116) and the free Hamiltonian

$$H = \frac{p_1^2 + p_2^2 + \ell p_1^2 p_2}{2m}$$

in equation (134). Section 6 realizes the deformed position algebra on $L^2(\mathbb{R}^2, d^2p)$. Finally, Section 7 considers two proposed composition laws for a pair of particles and associates each with a momentum-dependent deformation of the harmonic interaction.

The paper addresses a real conceptual difficulty: an abstract generator with a nonprimitive coproduct is not automatically a quantum observable, and the separation between abstract and canonical generators is potentially useful. The manuscript is also commendably explicit. Equations (89)–(102), (110)–(152), and the two-particle formulas make many of the claims directly testable. Extending the spatial κ -plane discussion to boosts and time translations and then following it into an interacting model could constitute a worthwhile contribution. At the displayed order, the canonical Lorentz brackets, the Casimir (99), the Galilean contraction, the one-particle realization, and the proper-dS interaction (178) are mutually consistent; the problems below therefore do not amount to a wholesale failure of the algebraic program.

In its present form, however, the central claims are not yet established to the standard required for publication. Most decisively, the second of the two interacting models is algebraically inconsistent as printed: the potential in equation (189) does not commute with the proposed total rotation generator and therefore violates equation (187). Independently, the new Hopf algebra in equations (45)–(59) is asserted with its derivation deferred to future work; the claimed generalized Noether result depends on an unproved parameter ansatz and begins by assuming an inner unitary implementation; and the multiparticle construction assumes an ordinary tensor product without addressing braiding, exchange, or the extension to three particles. The first-order model is also treated as an exact theory on the whole momentum plane even though equation (134) is then unbounded below.

I therefore recommend **major revision**. The underlying program is interesting enough that rejection would be premature, but publication should require a corrected two-particle calculation, an auditable derivation of the new symmetry structure, a controlled perturbative interpretation, and a substantially more complete treatment of the multiparticle sector. The claims of a fully relativistic interacting model and of general or unique constructions must also be narrowed unless the missing results are supplied.

Major comments

1. The potential in equation (189) fails the rotational-invariance condition (187).

This is a direct first-order contradiction within Section 7. Let

$$\Delta x_j = x_j^A(0) - x_j^B(0)$$

and use the manuscript's single-particle realization

$$x_1^I(0) = q_1^I - \ell p_2^I q_1^I, \quad x_2^I(0) = q_2^I,$$

together with the proposed coproduct-inspired total rotation generator

$$R = r^A + r^B - \ell p_2^A r^B$$

from equation (163). Substitution of the potential V_κ^{AB} printed in equation (189) gives

$$[R, V_\kappa^{AB}] = \frac{i\ell g}{2} (x_1^B(0) - x_1^A(0)) + O(\ell^2),$$

rather than zero. This residual is already present at the level of the canonical commutation relations and is nonzero for a generic two-particle state.

The uncanceled term comes solely from $\ell g[x_2^A(0) - x_2^B(0)]/2$ in equation (189). The remaining order- ℓ terms cancel the rotation of the undeformed oscillator potential against the order- ℓ part of R . Operator ordering cannot repair the displayed result: the products in equation (189) are symmetrized, the relevant zeroth-order generators are at most quadratic, and the Weyl commutator here agrees with i times the corresponding Poisson bracket. Simply deleting the offending linear term appears to restore all three conditions (186)–(188) at this order, but the authors should not rely on this observation without rederiving the full solution of the ansatz (166).

This error invalidates one of the two advertised “consistent” interacting descriptions and directly affects the abstract, introduction, and conclusions. The revised paper should display the coefficient equations obtained from (166), give their general solution, and explicitly list all commutators of the corrected total generators with the corrected Hamiltonian.

2. The proposed $\mathrm{TC}\kappa$ Hopf algebra, a main novelty of the paper, needs to be derived and fully checked in this paper.

Section 3 presents the coproducts (45)–(48), antipodes (49)–(52), and commutators (53)–(59), but the bottom-up construction is deferred to a future publication. The text says that coassociativity, compatibility with the coordinate algebra, and the coproduct-homomorphism property are “easy” or “straightforward” to verify. These assertions are not an adequate substitute for a derivation when this algebra is the foundation for every later result.

At minimum, the manuscript should provide the covariance calculation for the coordinate relations (6)–(7), all Jacobi identities, coassociativity, the left and right antipode checks, the counit, and the homomorphism checks represented schematically by equation (60). Since the physical discussion requires self-adjoint generators, a compatible $*$ -structure must also be specified and checked. The assumptions that fix the chosen momentum basis should be stated, and the authors should determine whether the algebra is related by a nonlinear basis change or twist to a known Lie-type deformation.

All these statements are made only through $O(\ell)$. First-order closure does not by itself prove the existence of an all-order Hopf algebra. The repeated descriptions “full characterization” and “fully consistent” should therefore be replaced by “first-order” or “perturbative” unless an all-order completion is supplied. The same qualification is needed for the canonical algebra, its contraction, and the two-particle models.

3. The logical status of the claimed generalized Noether theorem must be clarified.

Equation (35) begins by assuming that a symmetry is an inner unitary automorphism generated by a self-adjoint operator commuting with the constraint. This is already the observable implementation that the paper ultimately seeks to infer. The passage from equations (37) to (40) then depends on the momentum-dependent linear representation (38)–(39) of the noncommutative transformation parameters. The manuscript explicitly calls this representation a reasonable assumption and proves it only in the examples at hand. Consequently, equations (40)–(42) give a useful conditional representation statement, not a general Noether theorem for noncommutative spacetimes.

The authors should formulate a precise proposition: specify the coordinate algebra, differential calculus, representation, domains, constraint, and parameter ansatz; state whether inner implementability is a hypothesis or a conclusion; and prove the resulting statement under those hypotheses. For the $TC\kappa$ model, the concrete realization (74)–(102) can then serve as an existence example. Claims that the result applies to “most, if not all” physically interesting noncommutative spacetimes are not supported by the two examples discussed and should be removed or proved.

There is a related issue in the covariant-quantization discussion. For a continuous mass-shell constraint, the kernel of $p^2 - m^2c^2$ in the kinematical L^2 space is normally trivial; physical states live in a distributional completion and acquire an inner product by group averaging or an equivalent construction. One must also select the relevant energy sector to obtain the usual irreducible spin-zero representation. The simple statement in the Introduction that the physical Hilbert space is the kernel of the constraint is therefore insufficient, particularly when later arguments rely on self-adjoint operators descending to that space. The cited refined-algebraic-quantization framework should be implemented rather than only mentioned.

4. The ordinary tensor-product assumption in equation (153) is a major unproved input.

The proposed abstract symmetry has a noncocommutative coproduct, yet Section 7 assumes without discussion that the two-particle observable algebra is the ordinary tensor product of two commuting single-particle algebras. In a Hopf-symmetric theory one may instead need a braided tensor product, nontrivial cross-relations, an R -matrix, or deformed statistics. Any of these choices can change the total generators and the interaction. This is not a peripheral issue: the paper’s main physical claim concerns precisely the relation between the one- and two-particle symmetry structures.

The coproduct-inspired law (161)–(163), (179)–(184) is ordered in A and B and the potential (189) is not invariant under their exchange. The authors should state whether the particles are distinguishable, explain the role of the opposite coproduct, and show how identical-particle states would be defined. They should also show that the mass-dependent correction in equation (184) arises from a universal coproduct after adjoining mass as a central generator, or else describe it only as a special two-body representation.

Finally, a proposed composition law should be associative when applied to three particles, and a physical interaction should have an intelligible cluster limit. Neither property is examined. A three-particle test and a discussion of exchange and cluster behavior are necessary before the two-particle formulas can be interpreted as a multiparticle realization of the symmetry.

5. The two examples do not yet establish a physical composition-law–interaction correspondence.

The large coefficient ansatz (166) is followed immediately by particular potentials (178) and (189). The linear system for the coefficients is not shown, homogeneous invariant additions are not classified, and uniqueness is not claimed or proved in a controlled function class. Thus Section 7 establishes, at best, the existence of candidate interactions for two chosen generator maps. It does not show that a composition law determines an interaction, nor that the displayed interactions exhaust the possibilities.

There is also a concrete basis-equivalence issue. To first order, the “proper-dS” momentum law (158)–(159) is made additive by the single-particle change of variables

$$\tilde{p}_1 = p_1 + \ell p_1 p_2, \quad \tilde{p}_2 = p_2 - \frac{\ell}{2} p_1^2.$$

Indeed, if $\oplus$ denotes (158)–(159), then

$$\tilde{p}(p \oplus q) = \tilde{p}(p) + \tilde{p}(q) + O(\ell^2).$$

Therefore the nonlinear law alone does not demonstrate a physically distinct momentum geometry at the order considered. If de Sitter curvature is meant to be invariant content, the relevant metric, connection, and curvature should be given. More generally, the authors should identify an invariant observable, spectrum, transition amplitude, or dynamical effect that distinguishes the two constructions from each other and from a nonlinear reparametrization of ordinary variables.

The notation surrounding equations (168)–(169) also needs correction: H is first set equal to $h^A + h^B$, while equation (174) later defines the actual total generator by adding V_{dS}^{AB} . The composition law for time translations must be stated consistently.

6. The first-order expansion is used outside its regime of validity, and the operator theory is not controlled.

Equation (134), if treated as the exact Hamiltonian on the full space $L^2(\mathbb{R}^2, d^2p)$ claimed in Section 6, is unbounded below for every nonzero ℓ : the cubic term dominates with negative sign along one direction of large p_2 . The interacting Hamiltonians contain additional cubic momentum-position terms with the same problem. This does not necessarily invalidate a low-energy effective expansion, but it does rule out the manuscript’s global interpretation of the truncated formulas as a complete quantum model.

The paper should impose and track a regime such as $|\ell p| \ll 1$, state whether the model is a formal deformation or a regulated effective theory, and specify how time evolution is defined without extending the truncated Hamiltonian into its unstable large-momentum region. If a cutoff is used, its compatibility with the deformed position and boost operators must be addressed. If instead an all-order Hamiltonian is intended, it must be constructed and shown to be bounded appropriately.

The claims of self-adjointness also require common invariant domains for the unbounded generators and Hamiltonians. Schur’s lemma does not by itself show that an irreducible symmetry algebra

“coincides” with the full algebra of observables, and the Stone–von Neumann step needs the regular Weyl form of the canonical relations. The singular inverse implicit in $x_1 = (1 - \ell p_2)q_1$ at $p_2 = 1/\ell$ is another indication that the global and perturbative interpretations must not be mixed.

7. The scope and relativistic claims should match what is actually constructed.

The Poincaré-covariant construction through Section 4 is a single-particle one. The interacting two-particle model is introduced only after the $c \rightarrow \infty$ contraction and is covariant under a deformed Galilei algebra. It is therefore inaccurate to advertise a “fully relativistic first-quantized model of two particles propagating and interacting” on the noncommutative spacetime. What is presently obtained is a first-order, Galilei-covariant interacting model motivated by a relativistic single-particle construction. This distinction should be made explicit in the abstract, Introduction, Section 7, and conclusions.

The discussion of the Currie–Jordan–Sudarshan result around equations (154)–(157) should also be revised. The no-interaction theorem does not simply state that every generator of an interacting relativistic particle system is additive. Under its world-line and canonical assumptions, nontrivial interactions are obstructed; in other formulations interaction terms enter the Hamiltonian and boosts. This is especially relevant because the theorem is used to motivate the Galilean restriction.

8. The spinless planar sector and the various uniqueness claims need sharper assumptions.

The footnote following equation (135) says that spin in 2+1 dimensions is trivial and uninteresting. This is incorrect: planar systems admit anyonic spin, and the planar Galilei group has a second, “exotic” central extension for which the boosts need not commute. Both facts are potentially relevant to noncommuting spatial coordinates. The authors are free to study the spin-zero, non-exotic sector, but they should describe this as a deliberate restriction and explain how it affects the claimed generality.

Likewise, the assertions that equations (89)–(91) give the unique canonical Lorentz generators, that equation (99) is the only momentum invariant, and that equation (143) gives the only deformed position observables are too strong without a stated ansatz. Functions of an invariant remain invariant; commutators with coordinates determine generators only up to terms in the relevant commutant; and momentum-dependent $O(\ell)$ additions can preserve some of the position requirements. The authors should state polynomial degree, dimensional, ordering, boundary, and commutative-limit assumptions and solve the corresponding general problem. In particular, the requirement that physical initial positions obey the background coordinate relation (140) is introduced as a reasonable postulate, not derived from the canonical symmetry algebra, and should be labeled as such.

9. Novelty and significance require broader positioning and a physical output.

The immediate comparison for the interacting construction is almost entirely with the 2025 predecessor cited as Ref. 14. Before “first” or “full” claims can be evaluated, the manuscript should engage more substantially with classifications, twists, classical r -matrices, and basis equivalences for Lie-type Poincaré deformations; multiparticle κ -Poincaré representations, braided tensor products, deformed statistics, and relative locality; planar Galilei symmetry and its exotic extension; and earlier quantum-mechanical oscillator models on Lie-type noncommutative spaces.

The cited work on braiding makes the silence about the tensor product in (153) particularly conspicuous.

At present the paper stops after constructing formal generators and potentials. Even one controlled calculation of an energy shift, normal-mode frequency, center-of-mass/relative-coordinate effect, or transition amplitude would materially strengthen the significance claim and help decide whether the two composition laws encode physics or only basis choices. Such a result should be evaluated within the same $|\ell p| \ll 1$ regime demanded by the first-order expansion.

Minor comments

1. The Hopf-algebra definition in Section 1 should include the counit. Equation (21) equals zero for the nonidentity generators because their counit vanishes; the general antipode identity is $m(S \otimes \text{id})\Delta(a) = m(\text{id} \otimes S)\Delta(a) = \epsilon(a) 1$.
2. The manuscript should distinguish more explicitly between the non-self-adjoint auxiliary spacetime coordinate $x_2 = q_2 - i\ell/2$ in Section 4 and the self-adjoint initial-position observable $x_2(0) = q_2(0)$ in Section 6. The notation currently invites the reader to identify two operators with different status.
3. Several first-order equalities are written as exact equalities. A uniform $+O(\ell^2)$ convention would make clear which commutators and composition laws have actually been checked.
4. The manuscript should state the dimensions of all coefficients in the potential ansatz (166), the assumptions used to reduce that ansatz, and whether the two masses and particles are meant to be arbitrary and distinguishable.
5. The statement that every nonprimitive coproduct necessarily produces nonlinear commutators is too broad; twist deformations can retain an undeformed algebra sector. The introductory review should be narrowed to the class of deformations actually used.
6. A final technical appendix collecting the complete first-order checks of equations (45)–(59), (89)–(102), (110)–(133), and both composite realizations would greatly improve reproducibility.