Referee Report

arXiv:2607.22938

"Mass/electric versus NUT/magnetic charges: duality from scattering amplitudes in $D \geq 4$ and for all bosonic spins"

Summary of the paper

The manuscript studies the Kerr–NUT family and its scalar, Maxwell, and linearized higher-spin analogues from a combined position-space and scattering-amplitude perspective. In Section 2 the authors rewrite the Chen–Lü–Pope family in Cartesian multi-Kerr–Schild form. The mass monomial and the several NUT monomials are associated with distinct roots $r^2_{(\alpha)}$ of the spheroidal radial equation (6). This organization also gives scalar, single-copy, and linearized multi-copy fields. In odd dimensions a partial-fraction identity makes the equal-NUT scalar equal to minus the mass scalar, equation (27), and one charge is redundant. In even dimensions the authors instead propose the nontrivial relation

$$\frac{J_{(D-5)/2}(\chi)}{\chi^{(D-5)/2}} \quad \longleftrightarrow \quad i(-1)^{D/2}\frac{J_{-(D-5)/2}(\chi)}{\chi^{(D-5)/2}},$$

as summarized in equations (31) and (32). This maps the mass/electric sector to the sector in which all NUT/magnetic charges are equal.

Section 3 translates the stationary fields into three-point source amplitudes. Starting from the known Myers–Perry expressions, the authors propose the equal-NUT amplitudes (74) and (75) by applying the same spin-raising operation to the Bessel-reflected scalar seed. They then introduce the formal operator $\mathcal{S}_\mu$ in equations (92)–(94) and a combinatorial expression, equation (97), intended to generate the amplitude for any symmetric massless bosonic spin. Section 4 maps these amplitudes back to de Wit–Freedman curvatures. Equation (115) is a compact all-spin curvature formula in terms of a tower of Bessel potentials. The authors compare it numerically with curvatures computed from the multi-Kerr–Schild fields through finite values of D and spin, and give additional checks in the equal-rotation limit. Appendix C studies the infrared convergence of the gauge-invariant equal-NUT curvatures.

Finally, Section 5 considers leading $2 \mapsto 2$ scattering between two sources. In four dimensions the two current identities in equation (133) make the amplitude vanish when both sources have self-dual or anti-self-dual charges. The authors report that the four current contractions in equation (132) are independent in higher dimensions, so the corresponding trivial-scattering mechanism does not persist.

Assessment of correctness and clarity

The manuscript has a coherent overall architecture, and several checks are persuasive. In particular, the odd-dimensional redundancy follows cleanly from the root representation; the four-dimensional reductions reproduce the expected trigonometric and helicity structures; and the distinction between the full equal-NUT sector and the "little duality" of the equal-rotation degeneration is handled thoughtfully. Appendix C also correctly identifies why derivatives entering a curvature improve the small-ξ behavior relative to the corresponding potentials.

Nevertheless, the central claims are not yet established at the level suggested by the title, abstract, and conclusion. Equation (31) is supported by numerical Fourier-space tests only through $D = 8$, while the related equal-rotation formula (43) is explicitly conjectural. The arbitrary-spin construction is itself introduced as a guess and is compared with the multi-Kerr–Schild curvatures only through spin four and $D \leq 8$. Moreover, the equal-NUT potentials are distributionally singular, and the paper does not supply the finite-part prescription required for many of them. These are not merely presentational omissions: equations (31), (74), (75), (94), (97), and (115) carry most of the claimed new physics.

There is also a concrete reality-phase problem. Equation (20) implies that an even-dimensional geometric NUT parameter has a factor of $-i$ relative to a real physical charge. The relevant metric perturbation must therefore be tested with the phase multiplying the Kerr–Schild monomial. Main-text equation (22) instead tests $\mathrm{Im}(\Phi_{(\alpha)}\ell^2_{(\alpha)})$, whereas Appendix equation (157) tests $\mathrm{Im}(i^{1-\epsilon}\Phi_{(\alpha)}\ell^2_{(\alpha)})$. These are inequivalent in even dimensions. The appendix expression has the phase expected from equation (20), so the main-text assertion and the associated numerical check must be revisited.

Novelty and significance

The proposed Bessel-order reflection is original and potentially useful. If proved and placed on a firm distributional footing, it would provide a striking link among higher-dimensional Kerr–NUT geometry, classical double/multi-copy structures, and source amplitudes. The compact spin-raising organization would also be valuable if it can be defined and derived for arbitrary spin.

Some other aspects are more incremental. The roots $r^2_{(\alpha)}$ are identified in Appendix A.4 with the known Chen–Lü coordinates and with eigenvalues of the principal tensor. The new element is therefore the Cartesian interpretation of the individual roots as mass and NUT Kerr–Schild monomials, rather than a new Kerr–NUT solution. Similarly, the mass-sector amplitudes build directly on the recent Myers–Perry results cited in Section 3.2. The manuscript should make this division between reformulation and genuinely new result sharper.

In my view the core idea is sufficiently interesting for a journal such as JHEP or Physical Review D, but only after the generality and physical meaning of the main formulas are brought into line with the evidence.

Recommendation

Major revision. I would be willing to reconsider a substantially revised version. Publication requires either analytic derivations of the claimed all-D, all-spin relations or a systematic narrowing of the claims to the dimensions and spins actually established. The reality, charge, and distributional issues below must also be resolved.

Major comments

1. **Prove the Bessel-reflection relation or state its conjectural status consistently.**

   Equation (31) is the seed for the even-dimensional duality, and equations (73)–(77) simply transfer it to the amplitude language. The text states that (31) was checked numerically only through $D = 8$, while the equal-rotation analogue (43) was checked through $D = 10$ and is explicitly called a conjecture. A numerical pattern in a few dimensions is not yet a derivation of the asserted result for every even D. The residue representation preceding equation (31), together with the factorization (147), appears to offer a possible analytic route. The authors should use it, a contour argument, or a controlled Fourier transform to prove the identity with all branch and contact terms specified.

   If such a proof is not available, equations (31), (32), and (77) should be labeled as conjectures, and the abstract and conclusion should say exactly which D have been checked. A table separating proved identities, semi-analytic equal-rotation results, and numerical tests would help greatly. The same distinction should be applied to the odd-dimensional spin-two and higher-spin pure-gauge claims in equations (154) and (157), which are also supported only in selected dimensions.

2. **Define and derive the spin-raising operator.**

   Equations (92) and (93) specify the action of $\widehat{\mathscr{D}}_\xi$ only on a function of ξ. The commutator rule in equation (95) is then imposed so that two applications reproduce the graviton formula. It is not shown that these rules define a unique, associative operator on arbitrary products, that different orderings give the same answer, or that the resulting algebra satisfies the necessary consistency conditions. Equation (97) is consequently described in the text as a guess.

   The authors should formulate $\widehat{\mathscr{D}}_\xi$ as a genuine operator, or explicitly as a normal-ordering/contraction prescription, and derive equation (97) by induction or a generating function. They should then prove that equation (94) is transverse and reproduces the de Wit–Freedman curvature of $\Phi_{(\alpha)}\ell^s_{(\alpha)}$ for arbitrary s. Finite checks through spin four, plus a four-dimensional helicity pattern checked through spin six, do not establish the "for all bosonic spins" claim. Without an all-spin proof, the title, abstract, and repeated description of this operator as a main result should be narrowed.

   The physical terminology also needs care. For $s > 2$, the construction gives source form factors and free Fronsdal fields; the paper does not construct a consistent interacting flat-space higher-spin theory. Calling these objects three-point "scattering amplitudes" is acceptable only if this restricted, linearized meaning is stated prominently.

3. **Supply a distributional definition of the equal-NUT sector.**

   For even D, the scalar seed (73) is nonanalytic at small ξ, and the potentials (109) become increasingly singular with spin. Section 4.1 acknowledges that the potentials and the field representatives (117) generally require a finite-part prescription. Appendix C proves convergence only for the gauge-invariant curvatures, with all rotation parameters nonzero and away from $z = 0$. Thus the formulas are not yet defined on a substantial part of the parameter space on which the paper states them.

   The revised manuscript should give one explicit distribution or finite-part prescription and determine how alternative prescriptions change zero modes, contact terms, source support, and pure-gauge pieces. It should treat, or clearly exclude, $z = 0$, a vanishing rotation plane, coincident rotations, and the limit $a_i \to 0$. Equality of curvatures away from the source is not by itself equality of source amplitudes as distributions. The authors should also explain whether the resulting kernels represent localized particle sources, extended defects, or formal complex stationary sources. This point is especially important because Section 5 treats them as particles in a scattering problem.

4. **Reconcile the reality conditions and define the physical charges.**

   As noted above, equations (22) and (157) differ by the essential factor $i^{1-\epsilon}$. The authors should repeat the check with the full charge-weighted perturbation and state precisely which real and imaginary parts are pure gauge. The revised result should be given analytically where possible; otherwise the tested dimensions, coordinate patches, and branch choices should be documented.

   Equation (20) also denotes every N_α^{ADM} as an ADM charge but derives only a normalization by analogy with the mass. Higher-dimensional NUT charge, and a one-form "magnetic" charge in $D > 4$, need not have the same codimension or asymptotic interpretation as an ADM mass or electric point charge. The authors should provide or cite the relevant surface-charge/flux calculation, describe the asymptotic class and any Misner-string or other singular loci, and state the physical reality conditions directly in terms of real charges.

   There is a related parametrization issue in equation (59): $S_{\mu\nu}$ is defined as $J_{\mu\nu}/M^{\mathrm{ADM}}$, but the same tensor is used in the pure-NUT amplitudes (73)–(75), where the mass may vanish. The rotation tensor should instead be introduced as independent geometric data, or the limiting relation among angular momentum, mass, and NUT charge should be explained.

5. **Define the higher-dimensional duality and its eigenstates precisely.**

   Equation (77) includes the phase $i(-1)^{D/2}$, but the inverse transformation, the square of the transformation, and its action on the charge vector are not written. Section 5 nevertheless calls $\psi^{\mathrm{M}} \pm \psi^{\mathrm{N}}$ self-dual and anti-self-dual. The authors should define a normalized linear map on the pair of sectors, demonstrate whether it is an involution, and identify its eigenvectors after imposing the reality conditions. Otherwise "self-duality" in $D > 4$ is only a name for two chosen combinations and should not be presented as an intrinsic condition analogous to Hodge self-duality.

   This clarification would also sharpen the distinction, already noted by the authors, among a map on a restricted family of stationary solutions, a symmetry of the linearized equations, Hodge duality to mixed-symmetry fields, and an Ehlers-type transformation.

6. **Substantiate the $2 \mapsto 2$ conclusion for general $D > 4$.**

   The statement following equation (132) is that the four current contractions are linearly independent for every even $D > 4$ and generic on-shell kinematics. The only reported verification is numerical in $D = 6$, with the gravitational case mentioned but not displayed. The authors should either prove linear independence for general even D, or restrict the claim to the case actually calculated. At minimum the independent Bessel structures, allowed kinematics, relative orientations of the two rotation tensors, and exceptional subspaces should be stated explicitly. The gravity calculation should be shown if it is retained as evidence.

   In addition, a nonzero leading one-boson-exchange amplitude rules out the particular four-dimensional trivial-scattering mechanism. It is not a general proof that no integrable higher-dimensional sector exists. The abstract's cautious phrase "no evidence" is appropriate; stronger claims in Section 5 should be brought to the same level.

7. **Make the finite checks reproducible and align the strength of the claims with them.**

   Several important statements rely on numerical or symbolic checks: reality in $D = 4, 5, 6$, odd-dimensional redundancy in $D = 5, 7$, the scalar duality through $D = 8$, the little-duality formula through $D = 10$, the curvature relation through spin four and $D \leq 8$, and the scattering test in $D = 6$. No component choices, sample points, branch conventions, precision, residuals, or code are supplied. These checks cannot presently be independently assessed.

   The revised paper should include a concise verification protocol and make the scripts or notebooks available as supplementary material. This would not replace an analytic proof of the central all-D, all-spin claims, but it would make the finite evidence useful. The authors should also identify which equal-rotation zero modes were dropped in each comparison.

8. **Sharpen the comparison with existing literature.**

   The manuscript cites the relevant Kerr–NUT, principal-tensor, Myers–Perry-amplitude, double-copy, and higher-spin literature, but the specific novelty is sometimes left implicit. Since equation (165) identifies the radial roots with principal-tensor eigenvalues and Appendix A.5 maps them to the Chen–Lü coordinates, the statement that the root/charge association is new should be compared explicitly with those constructions. Likewise, the authors should explain exactly what the equal-NUT amplitude adds to earlier Kerr–NUT classical double-copy work, and how the nonanalytic $J_{-\sigma}$ seed fits into existing higher-dimensional three-point and multipole classifications.

Minor comments

1. In Section 2.1, the NUT range is written as $0 < \alpha < m - \epsilon$. Given the sums in equations (2), (12), and (13), this should be $0 < \alpha \leq m - \epsilon$.

2. In Appendix A.3, the odd-dimensional equal-rotation limit is said to lose $m - 2$ solutions. The generic odd-dimensional equation has m roots and the equal-rotation equation has one, so $m - 1$ roots are lost.

3. The phrase "generic spins" in Appendix A.1 should be "generic rotation parameters." More generally, using "equal rotations" rather than "equal spins" would avoid confusion with the field spin s.

4. The statement in Section 3.1 that the amplitude–metric correspondence is "exact" should be qualified. The three-point amplitude reconstructs the exact Kerr–Schild metric function because the metric is linear in its Kerr–Schild monomials; equation (55) itself is a linearized curvature formula and does not include generic nonlinear observables.

5. Equations (96) and (97), the central shorthand and all-spin formula, should be labeled for ease of reference. The same applies to the commutator equation (95).

6. In the four-dimensional discussion around equations (81) and (82), it would be clearer to introduce a real physical magnetic/NUT charge from the outset rather than repeatedly translating an imaginary geometric parameter. This would make the dyonic phase and the meaning of $M \pm N$ transparent.

7. Appendix B notes only in prose that the second polarization is complex conjugated in a helicity basis. The conjugation should appear explicitly in equation (170), because it is relevant to signs in the factorized $2 \mapsto 2$ amplitude.

8. The conclusion openly acknowledges that proofs of numerically checked relations remain future work. That is welcome, but the same status should be visible where those relations are first presented, especially in Sections 2.5, 3.3, and 3.5.