Referee Report

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"Krylov complexity and spectral density of BMN matrix model"

Summary of the paper

The manuscript applies state and operator Krylov methods to a two-coordinate bosonic reduction of the BMN matrix model. The reduced Hamiltonian in equations (2.5)–(2.6) contains parameters k, l, n chosen to reproduce a pulsating fuzzy-sphere reduction (PFS) and an integrable fuzzy-sphere reduction (IFS). In Section 2 the author chooses a separable Gaussian state, constructs the first three state Krylov vectors, and obtains low-order moments, Lanczos data, and orthogonal polynomials in the large mass-deformation limit. The observed powers of μ are then extrapolated to arbitrary moments and arbitrary polynomial degree.

Section 3 attempts the analogous construction for operator growth with the Liouvillian $\widehat{\mathcal{L}} = [\mathcal{H}, \cdot]$. The initial object is a Gaussian multiplication operator. The manuscript reports the first two off-diagonal Lanczos coefficients, the second and fourth moments, early-time Krylov amplitudes, and orthogonal polynomials. Section 4 Laplace transforms these early-time amplitudes, identifies the result with a resolvent, and uses its imaginary part to propose a spectral function. The poles of that expression are interpreted as long-lived, non-propagating collective modes, with analogies to diffusion and non-Fermi-liquid behavior. A counterterm is then introduced to obtain a unit-normalized "density of states." Finally, Section 5 computes an early-time Krylov variance and claims linear real-time growth of the Krylov entropy, which is presented as evidence for quantum chaos.

The topic is potentially worthwhile. A controlled comparison of Krylov data in distinct fuzzy-sphere reductions could be useful, and an actual operator spectral measure for a BMN-sector observable would be interesting. However, the operator Hilbert space used in the manuscript is not well-defined, its Lanczos construction contradicts the stated Liouvillian recurrence, and a short-time Taylor series is used outside its domain to infer infrared spectral physics. These issues invalidate the main results of Sections 3–5.

Assessment of correctness and logical structure

The state-space part contains some explicit low-order algebra that could serve as a starting point, although the initial state and normalization conventions require correction. The central operator-space part does not currently define a valid normalized inner product. Even if one provisionally regularizes that problem, trace cyclicity forces the first diagonal Liouvillian matrix element to vanish, whereas equation (3.17) assigns it the value 2μ. This contradiction propagates into the Krylov vectors, moments, polynomials, and amplitudes.

There is a second, independent obstruction in the spectral analysis. Equations (3.32)–(3.33) are explicitly short-time expansions. Their term-by-term Laplace transforms are therefore a finite large-frequency moment expansion of the resolvent, not the resolvent at small frequency. Analytically

continuing the resulting Laurent polynomial to $z = 0$ creates the very poles that are subsequently interpreted as physical infrared excitations. The proposed spectral function is not shown to be positive, and the regularization involves undefined products of distributions.

The entropy conclusion is likewise incompatible with normalized unitary Krylov evolution. Starting from a single Krylov site, the first probability is of order t^2, so the Shannon entropy behaves as $t^2|\log t|$ at asymptotically early time, not linearly in t. The manuscript obtains a linear term only after an unjustified Wick rotation that produces an exponentially growing amplitude and does not preserve the probability sum.

Novelty and significance

The potentially new physics is concentrated in the claimed spectral, collective-mode, and entropy results. Those conclusions are not supported by the calculation, so the manuscript does not presently establish a novel publishable result at the standard of JHEP or Physical Review D. The low-order moment and polynomial calculations are applications of standard Lanczos and orthogonal-polynomial relations and, by themselves, have limited scope.

The distinction from the author's closely related papers also needs to be made explicit. The bibliography includes "Krylov state complexity for BMN matrix model," "Krylov Complexity for Plane Wave Matrix Model," and a related holographic Krylov paper, and the present Sections 2–3 repeatedly invoke those works. A future version should give a precise result-by-result comparison, separating formulas already obtained there from genuinely new calculations. It should also state what goes beyond the general orthogonal-polynomial and memory-function framework of Mück and Yang and of Mück. Merely verifying generic recurrences for the first two Krylov levels is not an arbitrary-n result.

Recommendation

Reject in the present form. The recommendation is based on the correctness of the main claims, rather than on presentation. Repair would require choosing and normalizing a legitimate operator inner product, recomputing the operator Lanczos chain, obtaining controlled all-time or large-n information, and reconstructing a positive spectral measure. The entropy and chaos analysis would also have to be replaced. This amounts to a new calculation rather than a conventional major revision.

Major comments

1. **The operator Hilbert space and its norm must be defined consistently.**

   Equation (3.3) takes $\mathcal{O}_0 = f(\hat{x}, \hat{y})$ to be a Gaussian multiplication operator, while equation (3.7) uses the unweighted Hilbert-Schmidt product

   $$(\mathcal{O}_1|\mathcal{O}_2) = \mathrm{Tr}(\mathcal{O}_1^\dagger \mathcal{O}_2).$$

   On $L^2(\mathbb{R}^2)$, a nonzero multiplication operator of this form is not Hilbert-Schmidt. Its kernel contains a position delta function, and the diagonal trace contains the corresponding $\delta(0)$.

The finite Gaussian integral used in the manuscript is the norm of a wavefunction, not the Hilbert-Schmidt norm of the multiplication operator. Acting $\mathcal{O}_0$ on a generalized position eigenstate, as in equation (3.4), does not provide an isometry from operator space to state space.

This point cannot be treated as an overall normalization that later drops out. The operator moments and Lanczos coefficients depend on the chosen inner product. The author should instead specify, for example, a finite-dimensional regulator, a normalized thermal/Kubo inner product, or a trace-class initial operator such as a rank-one projector. The regulator must be retained until cyclicity, positivity, and normalization have been checked. Each such choice leads to a different calculation.

2. **The operator Lanczos construction contradicts trace cyclicity and its own recurrence.**

   For a Hermitian $\mathcal{O}_0$, the inner product stated in equation (3.7) gives

$$(\mathcal{O}_0|\widehat{\mathcal{L}}\mathcal{O}_0) = \mathrm{Tr}\big(\mathcal{O}_0\mathcal{H}\mathcal{O}_0 - \mathcal{O}_0^2\mathcal{H}\big) = 0$$

   by cyclicity of a regulated trace. This agrees with the assertion following equation (3.9) that the diagonal entries a_n vanish. It directly contradicts equation (3.17), which sets $(K_0|\mathcal{O}_1) = 2\mu$, and equations (3.49)–(3.51), which build the first polynomial from $(\widehat{\mathcal{L}} - 2\mu)/(2\mu)$. With zero diagonal, the first polynomial is proportional to $\widehat{\mathcal{L}}$, with no such shift.

   The same inconsistency is present in the basis algorithm. Equation (3.9) uses the zero-diagonal operator recurrence $A_{n+1} = \widehat{\mathcal{L}}K_n - b_nK_{n-1}$, but equations (3.16)–(3.21) reintroduce coefficients c_n along K_n. The author must decide which recurrence is intended and derive it from the chosen inner product. At present, the quoted b_1, b_2, fourth moment, and orthogonal polynomials do not follow from a single consistent Lanczos chain.

3. **The time-series amplitudes violate the parity and initial-value properties of a Hermitian operator Krylov chain.**

   With $\varphi_0(0) = 1$, $\varphi_{n>0}(0) = 0$, and equation (3.13), a zero-diagonal chain necessarily begins as

$$\varphi_0(t) = 1 - \frac{b_1^2t^2}{2} + O(t^4), \qquad \varphi_1(t) = b_1t + O(t^3), \qquad \varphi_2(t) = \frac{b_1b_2}{2}t^2 + O(t^4).$$

   In contrast, equation (3.33) contains an imaginary t^3 term and equation (3.36) gives $\varphi_2 = O(t)$. The former yields a nonzero third moment, directly contradicting the statement immediately below equation (3.36) that all odd moments vanish. For a Hermitian initial operator with the trace product, the autocorrelation is real and even.

   These order-counting constraints also fix the leading operator complexity and Krylov-position variance:

$$\mathcal{K}_{\mathcal{O}}(t) = b_1^2t^2 + O(t^4), \qquad \sigma_n^2(t) = b_1^2t^2 + O(t^4).$$

   If the manuscript's $b_1 = 2\mu$ were retained, both coefficients would be $4\mu^2$, independent of k at this order. The k-dependent coefficients in equations (3.44), (3.46), and (5.4) arise because the manuscript allows the second Krylov site to be populated at order t. They therefore cannot be correct.

4. **The state-space starting point and Gram-Schmidt conventions also require revision.**

The quadratic part of equation (2.17) has frequencies $\omega_x = \mu\sqrt{k}$ and $\omega_y = \mu$. Its normalized ground-state wavefunction is proportional to

$$\exp\Big[-\frac{\mu}{2}\big(\sqrt{k}\,x^2 + y^2\big)\Big],$$

not to the k-independent $\exp[-\mu(x^2+y^2)]$ of equations (2.18)–(2.19). The natural widths scale as $x, y \sim \mu^{-1/2}$, not as μ^{-1}. If the latter Gaussian is intended merely as a chosen trial state, that is permissible, but it should not be called the oscillator ground state or the maximally overlapping state without an explicit variational comparison. Since the moments and PFS/IFS comparison depend on this choice, they must then be interpreted as state-dependent quantities.

There is also a normalization mismatch between equations (2.27)–(2.28) and (2.31)–(2.32). The former determine c_1 using an unnormalized K_1, through $c_1 N_1 = (K_1|\Psi_2)$, while equation (2.31) subsequently redefines K_1 to include $1/\sqrt{N_1}$ and equation (2.32) uses the same c_1. For a normalized K_1, the subtraction coefficient is $(K_1|\Psi_2) = c_1\sqrt{N_1}$. The identical ambiguity occurs in the operator equations (3.16)–(3.21). The second Lanczos coefficient, second polynomial, and all results depending on them should be recomputed with one normalization convention.

For arbitrary order, equation (2.15) has a further problem. The three-term Lanczos theorem applies when the new residual is constructed from $\mathcal{H}K_n$. If one instead begins with $\Psi_{n+1} = \mathcal{H}^{n+1}\Psi_0$, as the manuscript does, an ordinary Gram-Schmidt step must in general subtract the projections onto all earlier vectors, not only K_n and K_{n-1}. The low-order checks do not prove the proposed recursion for general n.

5. **A short-time expansion does not imply a finite Krylov space or an all-time Laplace transform.**

   The discussion after equation (1.1), after equation (2.38), and following equation (3.36) repeatedly infers that the Krylov subspace is finite because the approximation is restricted to $\mu t < 1$. This conclusion does not follow. Every smooth unitary Krylov evolution has $\varphi_n(t) = O(t^n)$ near the origin, whether its Krylov chain is finite or infinite. Finite Krylov dimension requires termination of the chain, such as $b_D = 0$, or a finite cyclic spectral support. No such result is shown here.

   This confusion is essential to Section 4. Equation (4.1) is evaluated by inserting the cubic Taylor polynomials (3.32)–(3.33) and integrating them from zero to infinity. The exponential kernel localizes the integral near zero only when $\Re z$ is large compared with the dynamical scales. In the retarded limit used later, the real regulator is taken to zero and long-time data are maximally important. The claim that the integral is "exactly equal to the value evaluated at $t = 0$" is false; at most an initial-value theorem relates $\lim_{z\to\infty} z\mathfrak{c}_0(z)$ to $\varphi_0(0)$.

6. **The proposed resolvent is only a large-$|z|$ moment expansion and contains no controlled infrared information.**

   For a normalized operator state, the exact resolvent in equation (4.35) has the asymptotic expansion

$$(\mathcal{O}|(z-\widehat{\mathcal{L}})^{-1}|\mathcal{O}) = \frac{1}{z} + \frac{m_1}{z^2} + \frac{m_2}{z^3} + \frac{m_3}{z^4} + \cdots.$$

   Equation (4.29) is precisely a truncation of this expansion:

$$\mathcal{Q}_0(z) = \frac{1}{z} + \frac{4\mu^2}{z^3} - \frac{\mu^3(k+25) + 2(l+3)}{z^4}.$$

It is not an exact resolvent near $z = 0$. In fact, its nonzero z^{-4} coefficient is another manifestation of the spurious odd moment discussed above.

Consequently, equations (4.37)–(4.58) do not establish a low-frequency spectral function, quasiparticle residue, lifetime, or density of states. To address these questions the author needs the full autocorrelation at long times, a controlled infinite continued fraction from sufficiently many Lanczos coefficients, exact diagonalization with a specified regulator, or another resummation whose convergence and spectral positivity can be demonstrated. No analytic continuation of a finite high-frequency series can replace that information.

7. **The spectral "divergences," counterterm, and physical interpretations are not well-defined.**

   For a self-adjoint Liouvillian and a normalized operator state, the resolvent is a Stieltjes/Herglotz transform of a nonnegative, normalized spectral measure. A finite inverse-power expansion corresponds, on the real axis, to delta functions and their derivatives in a distributional large-frequency expansion. It cannot be separated into ordinary $1/\omega^3$ and $1/\omega^2$ infrared spectral densities. In particular, the product $\delta(\omega)/\omega^3$ used in equations (4.54)–(4.57) is not a defined distribution. Adding another copy of that undefined object cannot prove the unit sum in equation (4.58).

   Several physical claims also attribute dynamics to the regulator. The parameter ϵ implements the boundary value of a resolvent; it is not a computed scattering rate. The poles $\omega = \pm i\epsilon\sqrt{5}$ disappear with this regulator and do not establish a relaxation time; moreover, one member of this pair lies in the upper half-plane and is incompatible with a retarded relaxation pole. Diffusion would require a conserved density and a momentum-dependent pole such as $\omega = -iDq^2$, neither of which is present in this two-degree-of-freedom quantum mechanics. No self-energy is calculated, so the factor called an imaginary self-energy cannot support the asserted $\tau \propto \omega^{-3}$ lifetime. Likewise, pointwise vanishing away from a discrete pole as $\epsilon \to 0$ is not evidence for strong ultraviolet damping.

   The quantity $\mathcal{A}(\omega)$ associated with one chosen operator is an operator spectral measure or dynamical structure factor, not the total density of states of the BMN model. Its integral equals $(\mathcal{O}|\mathcal{O}) = 1$ by normalization. Thus a representation-independent unit integral, once correctly derived, would be a sum rule rather than new spectral information. The language of Fermi liquids, quasiparticle weight, topological-insulator boundary modes, and transport should be removed unless a fermionic response, conserved quantities, and an appropriate kinematic limit are explicitly introduced.

8. **The Krylov entropy does not grow linearly at asymptotically early real time, and the chaos claim is unsupported.**

   From the exact initial-value structure,

$$p_1(t) = b_1^2 t^2 + O(t^4), \qquad p_0(t) = 1 - b_1^2 t^2 + O(t^4), \qquad p_{n\geq 2}(t) = O(t^4).$$

   Substitution into the definition (5.5) gives

$$S_{\mathcal{O}}(t) = b_1^2 t^2 \left[1 - \log\left(b_1^2 t^2\right)\right] + O(t^4 |\log t|),$$

   not equation (5.9). The rescalings and Wick rotations between equations (5.5) and (5.9) yield an exponentially growing real-time amplitude while φ_0 is set to one. This does not preserve $\sum_n |\varphi_n|^2 = 1$, and a Shannon entropy cannot be analytically continued in this manner.

Low-order early-time data are generic and do not diagnose chaos. One needs, for example, controlled large-n Lanczos growth, spectral statistics, OTOCs, or an intermediate-time regime with a demonstrated relation to a Lyapunov exponent. The manuscript itself acknowledges this in its final future-directions paragraph. There is also a direct conceptual tension: the $k = 1, l = 3, n = 3$ reduction is introduced as the integrable fuzzy-sphere model, yet essentially the same proposed entropy law is said to prove chaos for both reductions. Moreover, the strict large-μ, near-vacuum limit is dominated by decoupled harmonic oscillators. The chaos regime must be specified and benchmarked against the cited BMN mixed-phase-space literature rather than inferred from universal short-time behavior.

9. **The scope and arbitrary-order claims should be narrowed and supported.**

   The calculation concerns a two-coordinate bosonic reduction, with a special initial state and only the first few Krylov levels. It does not determine the spectral density or transport of the full BMN matrix model, including its generic matrix degrees of freedom, gauge-sector structure, fermions, or large-N spectrum. Any future manuscript should state clearly which subsector and observable are being studied and justify how its spectral measure relates to the full theory.

   The physical description of the large-μ limit should also be revised. Under the canonical scaling $x = X/\sqrt{\mu}$, $y = Y/\sqrt{\mu}$, the kinetic and quadratic terms are of order μ, whereas the cubic terms are of order $\mu^{-1/2}$ and the quartic terms are of order μ^{-2}. Thus the particular near-vacuum limit studied here becomes increasingly harmonic and weakly anharmonic. A large mass deformation is not by itself a strong many-body coupling, a high scattering rate, or a chaos mechanism.

   The all-order statements are also not derived. Equations (2.46)–(2.47) leave an undetermined function $f(a_n, b_n)$; equation (2.63) is a dimensional polynomial ansatz inferred from degrees one and two; and the large-μ Laplace formulas introduce unspecified coefficients ζ_n, χ_n. An arbitrary-n claim requires a proof, a closed recurrence with controlled asymptotics, or substantial numerical evidence. Otherwise the abstract and conclusions should describe these expressions as conjectural scaling patterns from low order.

Minor comments

1. Equations (2.44)–(2.45) have a sign inconsistency. Under the manuscript's definition $\mathcal{M}_2 = d^2\mathcal{R}/dt^2|_0$, the analytic coefficient is negative, but the PFS and IFS values in equation (2.45) are printed as positive. The factors of i should be standardized; alternatively, define conventional real spectral moments $\langle \mathcal{H}^n \rangle$.

2. The leading spread complexity from a state initially at K_0 is $b_1^2 t^2 + O(t^4)$. The a_0, a_1, b_2 dependence displayed at order t^2 in Section 2 should be rechecked against the exact initial-value recurrence.

3. The Laplace and retarded-frequency conventions are mixed. The integral with e^{-zt} requires $\Re z > 0$, whereas the text first sets $z = \omega + i\epsilon$ and later uses $\mathfrak{c}_0(\epsilon - i\omega)$. One convention should be fixed before performing analytic continuation.

4. The TeX source assigns the same label to two different equations numbered (4.41) and (4.43), making cross-references ambiguous. One bibliography key is also duplicated. These generate compilation warnings and should be corrected.

5. Several symbols are introduced without definition, including the α multiplying the displayed $b_4\mathfrak{c}_4$ relation and the ζ_n, χ_n of the alleged large-n generalization. The notation $b_1^2(u)$ appears to be a typo for $b_1^2(\mu)$, and a generic-k polynomial unexpectedly carries a PFS subscript.

6. Statements that formulas are "exact in μ" should be separated from leading large-μ results. In particular, a unit spectral sum rule cannot make the truncated low-frequency expression exact at finite μ.

7. Given the length of the algebraic expressions, the author should provide ancillary symbolic code or an appendix that permits checks of normalization, orthogonality, moments, and the PFS/IFS specializations. Withholding the full N_2 expression while relying on it downstream makes the calculation difficult to verify.

8. The terminology should distinguish the mass deformation μ from a coupling, the reduced quantum-mechanical spectral measure from a many-body density of states, and cumulative spectral weight from a density. The manuscript also requires a careful language edit, but this should follow the substantive reconstruction.