

# Referee Report

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*“Kaluza-Klein Perturbation Theory from Exceptional Field Theory”*

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## Summary of the paper

The manuscript develops the linear perturbation theory of ten- and eleven-dimensional supergravity around backgrounds described by generalized Scherk–Schwarz reductions of  $E_{6(6)}$  exceptional field theory. Section 2 reviews the  $E_{6(6)}$  formulation, the section constraint, the tensor hierarchy, and the embedding of five-dimensional maximal gauged supergravity. The flattened internal derivative  $\mathfrak{d}_M$ , the embedding tensor  $X_{MN}{}^P$ , and the tensor  $Z^{MN}$  encode the generalized parallelization data.

Section 3 is the broadest new part of the paper. The authors retain arbitrary dependence of the fluctuations on the internal coordinates while factoring out the generalized frame. Equations (3.5), (3.7), (3.13), and (3.15) introduce the linearized vector field strength, the differential operator  $\mathcal{Z}$ , the scalar current, and the scalar-vector operator  $\Pi$ . Equation (3.18) gives the linearized internal and external gauge transformations. The tensor, vector, metric, and scalar fluctuation equations are then stated in equations (3.22), (3.24), (3.26), and (3.28), together with their variations into the background equations of motion. Unlike earlier AdS applications of exceptional-field-theory Kaluza–Klein spectrometry, these formulae allow nonzero background gauge fields, higher forms, and spacetime-dependent scalar data.

Section 4 specializes to backgrounds with vanishing one- and two-form field strengths and constant generalized metric, while allowing a general five-dimensional Einstein metric. The authors organize gauge-for-gauge parameters, gauge parameters, fields, equations of motion, and Noether identities into a chain complex. They introduce three Hodge-like operators in equation (4.28), construct the hatted combinations in equation (4.46) and the projectors in equations (4.47)–(4.48), and exhibit projection, inclusion, and homotopy maps through degree three. The intended physical interpretation is a systematic Higgs/Stueckelberg decomposition into gauge-redundant and gauge-invariant sectors. Section 4.5 applies the projectors to scalar, vector, tensor, and spin-two equations, leading to the mass operators in equations (4.97), (4.103), and (4.111) and to the Lichnerowicz-type spin-two equation following equation (4.121).

Section 5 applies the general fluctuation equations to the near-horizon geometry of extremal Kerr–Newman–AdS<sub>5</sub> black holes embedded in type IIB supergravity. The background is a fibration of a squashed three-sphere over AdS<sub>2</sub>. The calculation is restricted to  $U(2)$ -singlet “simple” representations that occur only once and hence are asserted not to mix with fields of another five-dimensional spin. The scalar reduction gives equation (5.32); tensors give the first-order equation (5.34) and the two-dimensional mass (5.39); and vectors give a pair of scalar modes. The authors test these modes against the charged AdS<sub>2</sub> BF bound (5.22). The most striking physical claim is that higher Kaluza–Klein levels produce a sequence of instability regions tangent to the BPS line at the points (5.49), while leaving small stability dents between successive tangencies.

## Assessment of correctness and clarity

The manuscript has a strong organizing idea and contains material that could become a useful reference. In particular, the on-shell gauge-variation identities following equations (3.22)–(3.28) are valuable structural checks. The explicit homotopy data and the summary table in Section 4 make the proposed Higgs decomposition unusually concrete. Recovery of the classic  $\text{AdS}_5 \times S^5$  masses and of the level-zero black-hole calculation gives additional evidence for the framework. If the accumulation of higher-level near-horizon instabilities is confirmed, it is physically interesting: it would show explicitly that a stable five-dimensional consistent-truncation sector can miss instabilities among ten-dimensional modes.

There are, however, several issues that presently obstruct a correctness assessment of the central results. The Hodge construction is based on unproved integration-by-parts and spectral assumptions for the flattened exceptional derivative. Several principal equations have inconsistent free indices. The  $\text{AdS}_2$  coordinate transformation and the displayed Whittaker solution are not mutually consistent. The tensor stability formula requires a signed first-order mass that is absent from the tables. Finally, the all-level stability statements are not accompanied by analytic discriminants, branching data, code, or a controlled description of how an infinite union was evaluated. These are not merely requests for additional exposition: they affect the projectors, the physical degrees of freedom, and the numerical stability regions.

## Novelty and significance

The work should be positioned as three related advances. First, Section 3 extends exceptional-field-theory Kaluza–Klein perturbation formulae beyond the  $\text{AdS}$ -type backgrounds with vanishing gauge fields and constant scalars treated in earlier spectrometry papers. Second, Section 4 generalizes the authors’ recent torus model of homotopy transfer to a nontrivial  $E_{6(6)}$  tensor hierarchy that includes scalar, vector, tensor, and spin-two Stueckelberg sectors. Third, Section 5 extends the level-zero five-dimensional near-horizon spectrum to selected simple modes at every ten-dimensional Kaluza–Klein level.

The classic  $\text{AdS}_5 \times S^5$  eigenvalues, the black-hole background, the charged  $\text{AdS}_2$  BF criterion, and the level-zero spectrum are established inputs rather than new results. Nor does the manuscript calculate nonlinear gauge-invariant variables or interaction vertices. The fair all-order claim is that a verified contraction of the free complex would provide input to an algorithm for transferring nonlinear brackets in future work. Similarly, the black-hole result concerns selected  $U(2)$ -singlet simple sectors at all levels, not the full ten-dimensional spectrum at any level. With these distinctions made precise and the technical issues below resolved, the paper would be of clear interest to the supergravity, exceptional-field-theory, and Kaluza–Klein communities.

## Recommendation

**Major revision.** The general perturbation framework and the proposed higher-level black-hole effect are sufficiently novel and interesting for a journal such as JHEP or PRD. I would be willing to reconsider the paper after the functional assumptions behind the homotopy projectors have been established, the core equations have undergone a systematic algebraic check, the charged- $\text{AdS}_2$

and tensor conventions have been corrected, and the all-level stability conclusions have been made reproducible. In its present form the manuscript is not yet ready for publication.

## Major comments

1. **The Hodge construction requires an invariant measure and spectral hypotheses that are not presently supplied.**

Equations (4.21)–(4.24) define internal inner products using a bare  $\int dy$  and integrate the flattened derivative

$$\mathfrak{d}_M = \rho^{-1}(U^{-1})_M{}^N \partial_N$$

by parts as though it were divergence-free. This is not automatic. For an internal density  $\mu(Y)$ , the needed condition is of the form

$$\partial_N [\mu(Y) \rho^{-1}(U^{-1})_M{}^N] = 0,$$

supplemented by compactness without boundary or boundary conditions that remove surface terms. Without this statement, the adjoints in equations (4.26)–(4.27) can acquire extra divergence terms. The issue then propagates to the operators  $\Delta, \Delta_V, \Delta_T$  in equation (4.28), to their Green operators, and to every projector and homotopy map that follows. If the no-trombone condition supplies the required density, the derivation should be shown explicitly.

There is a second, distinct assumption in equations (4.29)–(4.34). Positivity can identify the kernel of a well-defined  $D^\dagger D$ , but it does not by itself establish a discrete orthogonal spectrum, closed range, or a Hodge decomposition for the generally constrained and potentially non-elliptic operators  $\Pi$  and  $\mathcal{Z}$ . The authors should state the functional spaces and domains, establish the hypotheses under which  $K, K_V, K_T$  exist, and explain whether these are genuine inverses or Moore–Penrose pseudoinverses defined mode by mode. At minimum this structure should be verified explicitly for the  $S^5$  example. Until then, the homotopy transfer is a useful formal construction but not yet a general result for the class of backgrounds claimed.

2. **The principal fluctuation and mass formulae need a systematic coset and tensor-index audit.**

Several displayed equations are not well-defined as written. In equation (3.22), the index  $M$  on  $\mathcal{Y}^{\mu\nu M}$  is summed on the right-hand side while  $L$  is left free. Equation (3.26) uses  $\mu$  both as a free metric index and as the dummy index in  $D_\mu J^{\mu K}{}_N$ . Most importantly, equation (4.97), which is one of the paper’s headline projected mass operators, uses  $KL$  simultaneously as free and contracted indices. The intended expression appears to be something like

$$(\mathcal{M}_{\text{sm}})_{MN}{}^{KL} = \mathbb{P}_{MN}{}^{RS} (M_{(0)}^2)_{RS}{}^{TU} \mathbb{P}_{TU}{}^{KL}.$$

The vector equation immediately preceding equation (4.103) similarly has  $\widehat{a}^{\nu N}$  where the free index requires  $M$ . In Section 5, the field-strength definition preceding equation (5.42) ends with  $a_\nu{}^N$  although its left-hand side carries  $M$ , and its expansion of  $2D_{[\mu} a_{\nu]}$  appears to miss a factor of two in the minimal-coupling term. Equation (5.43) ends with  $(\mathcal{M}_V)^M{}_N a^{\mu P}$ , leaving  $N$  and  $P$  unmatched.

The scalar fluctuations also need to be defined explicitly as tangent vectors to  $E_{6(6)}/USp(8)$ . The manuscript treats  $m_{MN}$  as a symmetric tensor, but later uses that its current is Lie-algebra valued. The linearized generalized-metric constraint and the coset projector should

be built into the spaces of fields, equations, and mass operators. The authors should supply corrected formulae and perform an independent symbolic index and gauge-variation check of equations (3.22)–(3.29) and (4.97)–(4.123). Given the density of the expressions, a machine-readable notebook would be useful.

**3. The scope of the homotopy equivalence and of the “all-order” claim should be made precise.**

The explicit result is a contraction of a linear complex. Section 4.2 states that further gauge-for-gauge spaces almost certainly exist, while the new zero-mode transformation in equation (4.39) is itself motivated by a higher three-form field absent from the retained field space. The extension through equations of motion stops at degree three, and the manuscript notes after equation (4.95) that the next differential is not known. The nontrivial chain-map and homotopy identities in equations (4.78)–(4.95) are mostly attributed to “explicit computation,” but those calculations are not displayed. A contraction intended for nonlinear  $L_\infty$  transfer should also state the usual side conditions and verify  $p\iota = \text{id}$ , or explain precisely which weaker data are sufficient.

The authors should either embed the construction in the full linearized tensor hierarchy and prove that the omitted stages do not affect the cohomology relevant to the physical spectrum, or limit the theorem to the displayed truncated complex. They should also distinguish what is proved here from what the homotopy-transfer theorem could produce after the nonlinear brackets have been supplied. No nonlinear field redefinition or interaction vertex is transferred in the present paper. The conclusion should therefore say that the free contraction is a prospective input for an all-order algorithm, rather than that all-order physical fields and  $n$ -point couplings have already been determined.

**4. Two exceptional sectors in the restricted-background analysis need separate treatment.**

First, the passage from the projected tensor equation (4.11) to equation (4.107) is not evidently an equivalence. The hatted vector satisfies  $\mathcal{Z}_{MN}^\dagger \widehat{a}_\mu{}^N = 0$  by equation (4.49). After raising/lowering indices as in Section 4.2, the  $2\nabla^{[\mu} \widehat{a}^{\nu]}_M$  term is consequently in the kernel of the outer  $\mathcal{Z}$  in equation (4.11). The original equation constrains only the appropriate nonzero- $T$  projection of the bracket, whereas equation (4.107), written without that projection, appears additionally to set its kernel component to zero. The authors should retain the projector explicitly and show that no spurious condition such as  $d\widehat{a} = 0$  is imposed. The second-order tensor equation may survive, but the first-order derivation should be corrected.

Second, the spin-two trace argument leading to equation (4.121) assumes that  $\Delta + V/4$  is invertible. At a kernel of this operator the trace cannot be discarded and the degree-of-freedom count and gauge structure can change; this is precisely the partially massless exceptional case mentioned by the authors. Since the claimed result covers general Einstein backgrounds and is described as a complete mass spectrum, this possibility cannot be left as an aside. It should either be analyzed as a separate branch or explicitly excluded from the statement of the result.

**5. The central algebraic identities and mass-spectrum checks should be made independently reproducible.**

The nilpotency relation  $\Pi\mathcal{Z} = 0$ , used in equation (4.3) and throughout the chain complex, ultimately relies on the identity in equation (A.3). That identity is said only to have been checked in Mathematica with a generic parameterization of the embedding tensor. Because

it underlies  $\partial^2 = 0$ , the projectors, and the Higgs interpretation, an analytic representation-theoretic proof or the complete notebook and parameterization should be provided.

Likewise, the projected scalar and vector masses are said to reproduce the  $\text{AdS}_5 \times S^5$  spectrum at low levels, but no eigenvalues, multiplicities, or code are shown, and analogous tensor and spin-two checks are absent. The paper should display several complete Kaluza–Klein levels with multiplicities, identify the spurious zero directions removed relative to the earlier naive matrices, and verify equations (4.97), (4.103), and (4.111) against the known spectrum. An all-level argument would be ideal; otherwise an ancillary notebook is appropriate. This benchmark is especially important because it is both the main correctness test and the clearest demonstration of novelty relative to earlier exceptional-field spectrometry and to the authors’ torus model.

**6. The charged- $\text{AdS}_2$  normalization in Section 5.2 is internally inconsistent.**

Starting from equation (5.16) and setting  $z = (r - r_+)^{-1}$ , the standard form

$$ds_2^2 = \frac{B^2}{z^2}(dz^2 - d\tau^2)$$

requires  $\tau = (A/B)t$ , not  $\tau = (B/A)t$  as printed. With the correct coordinate, the invariant electric potential is  $A_2 = -B^2 d\tau/z$ , up to orientation, and its field strength is the  $\text{AdS}_2$  volume form. Equations (5.23)–(5.24) then cannot use the coordinate-dependent component  $A_{2,\tau} = -B^2/z$  as a constant Whittaker index or inside a constant conformal dimension. The relevant electric coefficient is  $qB^2$ , so that

$$\nu^2 = \frac{1}{4} + m^2 B^2 - q^2 B^4.$$

The BF bound (5.22) appears to contain this corrected combination, but the derivation does not.

The authors should repair the coordinate change and Whittaker solution and then rederive the effective charges in equations (5.32) and (5.38), the tensor/vector masses, and the plotted discriminants from one consistent normalization. The agreement of equation (5.22) with the expected result is encouraging, but it is not sufficient to establish that every subsequent factor of  $A$  and  $B$  was implemented correctly.

**7. The tensor and vector sectors do not contain enough consistent information to reproduce the black-hole stability calculation.**

The tensor equation (5.34) is first order, so the sign of its eigenvalue  $M$  is physical convention data. Equation (5.39) contains both  $Q/M$  and  $Q^2/M^2$ , yet Table 4 supplies only  $M^2$ . The signed  $M$  for each representation, and its correlation with  $Q$ , the  $SO(2)$  charge, and the conjugate tower, must be tabulated. The anti-Hermitian convention in equations (5.33)–(5.34) should also be explained. Without these data, the claimed tensor stability cannot be reconstructed.

For vectors, equation (5.42) contains  $D_\nu f^{\mu\nu}$ , whereas equation (5.43) replaces it by  $\nabla_\nu f^{\mu\nu}$ . For  $Q \neq 0$ , the latter is not gauge covariant; the separate  $iQ\mathcal{F}^{\mu\nu}a_\nu$  term does not cancel the derivative-of-gauge-parameter variation of  $\nabla_\nu f^{\mu\nu}$ . The covariant derivative should be retained or the cancellation should be demonstrated explicitly. In addition, equations (5.47)–(5.48) divide by  $G$ , although the parameter domain includes  $a = 0$ , where  $G = 0$ . A nonsingular expression or an explicit limiting derivation is required. The endpoint  $a = 1$  in equation (5.5) also needs clarification because several background coefficients in equation (5.4) diverge there.

**8. The simple-sector selection and the infinite-level stability claims need derivations and a reproducible computational record.**

The absence of a second field with the same displayed representation is a plausible way to identify diagonal sectors, but the necessary branching data are not given. The manuscript should show the  $SO(6) \rightarrow SU(3) \times U(1)$  branching with multiplicities, allowed ranges of  $n$  and  $k$ , and the reality and gauge identifications used to obtain equations (5.28)–(5.29) and Tables 4–5. It should also demonstrate representation by representation that the Pauli operator (5.31) vanishes for all simple scalars, that  $C^N \Pi_{MN,P}$  vanishes for the simple vectors, and which eigenvalue of  $C^K d_{MKL}$  is used in passing from equation (5.42) to equation (5.43).

Section 5.4 then makes genuinely infinite-level statements: all simple vectors and tensors are stable throughout the parameter space, all other simple scalar families remain outside the teal region, Figure 6 represents the whole  $t^+$  tower, and equation (5.49) gives the exact tangencies. The authors should provide the charged-BF discriminant for every family as an explicit function of  $n, k, a, r_+$ , prove the positivity and tangency claims analytically where possible, and explain whether the final union was analytic or evaluated to a finite cutoff. If a cutoff was used, its value and a convergence or tail bound are needed. The branching, diagonalization, BF-testing, and plotting scripts should be archived with unrounded data. The BPS-line check and recovery of the level-zero result are useful, but do not by themselves establish the infinite-tower conclusion.

**9. Near-horizon BF stability must be distinguished from stability of the complete black hole.**

Violation of equation (5.22) diagnoses an oscillatory or BF-violating mode of the charged  $AdS_2$  near-horizon problem. Whether that mode extends to a normalizable growing perturbation of the full extremal, asymptotically  $AdS_5$  black hole requires a matching argument with global boundary conditions, or a theorem whose hypotheses are verified for these charged Kaluza–Klein fields. Conversely, satisfying the near-horizon bound for the selected simple  $U(2)$ -singlet sectors proves neither global stability nor stability against the omitted nonsinglets and mixed-spin sectors.

The plots and conclusion should therefore consistently use language such as “near-horizon BF-stable region for the analyzed simple sectors.” The relation to full black-hole stability should be discussed explicitly. The abstract and conclusion should also replace “the full ten-dimensional Kaluza–Klein towers” with the more accurate “selected simple  $U(2)$ -singlet modes at every ten-dimensional Kaluza–Klein level.”

## Minor comments

1. In the scalar kinetic term of equation (2.10), the second generalized metric should carry lower indices. As printed, the same upper indices occur twice.
2. Equation (5.9) writes the  $SU(3)$  adjoint Dynkin label as  $[1.1]^0$ ; this should presumably be  $[1, 1]^0$ .
3. The statement after equation (5.14) that the scalar background equations are trivial because there are no  $U(3)$  singlets should be rephrased. Table 2 contains  $SO(6)$ -singlet scalar modes. If their nonzero  $SO(2)$  charge, or a larger retained symmetry, excludes them from the background truncation, that is the argument that should be stated.

4. The allowed ranges of the level labels should accompany Tables 2–5. Several displayed representations otherwise acquire negative Dynkin labels at the first nominal values of  $n$ .
5. The hatted spin-two and vector fields in equation (4.50) retain ordinary zero-mode gauge transformations. Calling every hatted field “physical” or “gauge invariant” obscures the remaining massless gauge redundancy. “Invariant under massive Stueckelberg transformations” would be more precise.
6. Saying that  $M_{MN} = \delta_{MN}$  means that “all scalar fields are zero” is imprecise. The generalized metric is at the scalar origin, not zero.
7. The manuscript would benefit from a careful copy edit. Examples include “anti-hermitean,” “estbalished,” “exlicit,” “approppriste,” and “The simplest case are.” Notation for background quantities, covariant derivatives, and raised/lowered  $E_{6(6)}$  indices should also be made uniform.