

Referee Report

Feynman Tree Theorem and the Gelfand–Yaglom Formula

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Recommendation: Major revision. The central one-dimensional identity appears correct, but the manuscript requires a complete proof of its key combinatorial step, precise determinant and boundary-condition hypotheses, correction of two matrix-kernel statements, and a substantially sharper account of what is new and of what is actually established about the Feynman Tree Theorem and quantum field theory.

Summary and overall assessment

The manuscript proposes a diagrammatic derivation of the Gelfand–Yaglom formula. For $\mathcal{O}_g = \partial_t^2 + \omega^2 - g u(t)$ on a finite interval, the authors write the free Dirichlet Green function as a retarded kernel minus a separable endpoint term. Inserting this decomposition in the trace-log expansion of the relative determinant eliminates purely retarded cycles and turns the remaining loop terms into products of retarded chains. These chains are then identified with the Volterra expansion of the initial-value solution $y_g(t_f)$. The construction is subsequently lifted to a first-order 2×2 system for general linear boundary conditions, leading to

$$\frac{\det \mathcal{O}_g}{\det \mathcal{O}_0} = \frac{\det \mathbb{M}_g}{\det \mathbb{M}_0}, \quad \mathbb{M}_g = M_i + M_f U_g(t_f, t_i).$$

There is a sound and potentially useful idea here. In particular, the signs and operator conventions in the Dirichlet calculation are consistent. If

$$T_m = \frac{y_g^{(m)}(t_f)}{y_0(t_f)}$$

denotes the normalized m -vertex Volterra chain, direct expansion gives

$$\begin{aligned} \text{Tr}(G_D u) &= -T_1, \\ \text{Tr}[(G_D u)^2] &= -2T_2 + T_1^2, \\ \text{Tr}[(G_D u)^3] &= -3T_3 + 3T_1 T_2 - T_1^3. \end{aligned}$$

Together with the factor $-1/n$ in equation (6), these are exactly the first three coefficients of $\ln[1 + \sum_{m \geq 1} g^m T_m]$. The matrix resummation in equations (22)–(25) is likewise algebraically correct once its sums and kernels are defined precisely. I therefore do not find a fatal error in the principal one-dimensional identity.

The present version nevertheless falls short of the standard needed for publication in a high-energy theory journal. The all-orders combinatorial statement that carries the new diagrammatic proof is asserted rather than derived, and the explanation of equation (9) is not correct as written. The functional determinant is not defined with sufficient precision, the free zero-mode condition needed already in equation (7) is omitted, and the general-boundary section contains a wrong operator in equation (17) and a distributional contact-term problem in equation (15). More conceptually, the paper has not yet shown that its finite-interval resolvent identity is a direct instance of the standard Feynman Tree Theorem, as opposed to a close loop-tree analogue. Finally, the general determinant formula and its Volterra reduction have substantial antecedents, while the proposed higher-dimensional payoff remains speculative.

Correctness and logical structure

The logical route is appropriate: equations (2)–(5) generate the initial-value trees, equations (6)–(7) generate the cut loop expansion, and the logarithm should identify the two series. The Dirichlet Green-function identity in equation (7) has the correct sign for an operator whose leading term is $+\partial_t^2$, and the plus sign in the Volterra equation follows from $\mathcal{O}_0 y_g = g u y_g$. The low-order checks above also confirm that no simple sign or symmetry-factor mistake is hidden in the construction.

The weakness is at the decisive transition from equation (9) to the claimed all-orders equality. That step must distinguish an ordered composition of a cyclic trace from a multiplicity partition, explain the choice of an initial cut, and account for cyclic stabilizers. Without this derivation, the second-order picture in equation (10) is evidence but not a proof. The general-boundary section later states the more useful ordered-composition formula in equation (22), but it also needs a derivation and an explicit statement that the $\mathcal{X}_m$ are generally noncommuting matrices.

Under the usual admissibility and invertibility assumptions, the final formula (25) is correct and is the standard one-dimensional Forman result. The manuscript’s potentially new contribution is consequently not that determinant formula itself, but a specific cut-diagram interpretation and its combinatorial realization.

Novelty and significance

Recasting a familiar spectral identity in diagrammatic language could be valuable to the high-energy community, particularly if it gives a concrete bridge to regulated field-theory calculations. The current manuscript does not yet demonstrate that broader consequence. Green functions of ordinary differential operators are semi-separable kernels, and the reduction of their Fredholm determinants to finite-dimensional determinants through Volterra equations is central to the cited work of Gesztesy and Makarov. General linear boundary conditions and the matrix formula in equation (25) are also covered by Forman-type and general Sturm–Liouville determinant theorems. A precise comparison is therefore essential: the authors should identify exactly which graphical identity, cut rule, or combinatorial statement is new.

The discussion of higher dimensions also needs revision in light of the cited work of Ossipov, which already formulates higher-dimensional Gelfand–Yaglom relations as matrix/operator initial-value problems in discrete and continuum settings. What may remain new is an *FTT-based interpretation* of such evolution, not the existence of a matrix initial-value formulation. As written, the manuscript is an interesting mathematical-physics note rather than a result with established PRL-level breadth. It could become suitable for a journal such as PRD or JHEP after a major revision that both secures the proof and delineates its physical novelty.

Major comments

1. Supply the all-orders cut/composition proof.

The main new step should be stated analytically. If K cuts divide an n -vertex cyclic loop into

ordered chains of lengths $\ell_1, \dots, \ell_K$, then the identity needed in the Dirichlet case is

$$\mathrm{Tr}[(G_D u)^n] = \sum_{K=1}^n (-1)^K \frac{n}{K} \sum_{\substack{\ell_1 + \dots + \ell_K = n \\ \ell_j \geq 1}} T_{\ell_1} \cdots T_{\ell_K}.$$

It immediately implies

$$\begin{aligned} - \sum_{n \geq 1} \frac{g^n}{n} \mathrm{Tr}[(G_D u)^n] &= \sum_{K \geq 1} \frac{(-1)^{K+1}}{K} \left(\sum_{m \geq 1} g^m T_m \right)^K \\ &= \ln \left(1 + \sum_{m \geq 1} g^m T_m \right). \end{aligned}$$

The manuscript should prove the first identity, including cyclic symmetry and repeated chain lengths, rather than infer it from the diagram in equation (10).

The prose surrounding equation (9) also needs correction. For $K = \sum_i m_i$, the coefficient produced directly by $\ln(1+x)$ is

$$\frac{(-1)^{K+1}}{K} \binom{K}{m_1, \dots, m_n}.$$

Equation (9) is $-n$ times this coefficient and belongs to the loop trace before multiplication by the $-1/n$ in equation (6). Moreover, its absolute value does not count cuts of an “unlabeled loop modulo rotations” in the simple sense claimed: the factor n records cut placements among the cyclic trace positions, while the factor $1/K$ removes the choice of a starting cut. The authors should replace the current counting statement with a precise bijection. An explicit third-order example would be an efficient and convincing check.

2. Define the determinant and state the analytic hypotheses.

Equation (6) naturally defines a relative Fredholm determinant $\det(I - gG_D u)$. The authors should say so, or instead specify a zeta-determinant prescription and a common spectral cut. In either case, they should state regularity assumptions on $u(t)$ sufficient for $G_D u$ to be trace class, and explain that the trace-log series initially converges only for sufficiently small g . The extension to finite g is then by analytic continuation away from zeros, with a chosen logarithm branch.

The free Dirichlet operator must be invertible before equation (7):

$$G_R(t_f, t_i) = y_0(t_f) = \frac{\sin[\omega(t_f - t_i)]}{\omega} \neq 0.$$

Otherwise equation (7) divides by zero and the ratio in equation (6) is not defined as written. Similarly, equation (23) fails at a value of g where $\det \mathbb{M}_g = 0$, although the determinant identity (25) can extend continuously and correctly vanish there. Zero modes, negative modes, and the determinant phase are physically important in the semiclassical applications cited in the introduction, so their status should not be left implicit.

For general boundary data the paper should require $\mathrm{rank}[M_i \ M_f] = 2$. If the intended operator is self-adjoint, the corresponding symplectic compatibility condition should also be stated. If non-self-adjoint boundary problems are included, the authors must instead specify the determinant and spectral assumptions for that setting.

3. Correct and clarify the matrix Green function.

Equation (17) must use the free matrix A_0 , not A_g . Both kernels in equations (15)–(20) are free kernels, and equations (18) and (21) explicitly use $\mathbb{M}_0$ and U_0 . The consistent equation is

$$(\partial_t I_2 - A_0) \mathbf{G}_{R/B,0}(t, t') = \delta(t - t') I_2.$$

There is also a distributional contact term in equation (15). Starting from the exact definition $\mathbf{G}_R(t, t') = \Theta(t - t') U_0(t, t')$, one finds

$$(\mathbf{G}_R)_{21} = -\partial_t \partial_{t'} G_R(t, t') - \delta(t - t'),$$

not simply $-\partial_t \partial_{t'} G_R$. The printed matrix is correct only away from the diagonal. The contact term may indeed drop out after the scalar projection and next to the nilpotent insertion $e_2 e_1^T$, but that cancellation should be shown rather than described as inconsequential. Defining the matrix kernel directly as ΘU_0 is the cleanest option.

Finally, in equation (18) the homogeneous factor is more transparently written as $U_0(t, t_i)$ rather than $\mathbf{G}_R(t, t_i)$. This avoids an artificial endpoint source and a dependence on the convention for $\Theta(0)$. The authors should derive the boundary identity explicitly and then show that the scalar projection in equation (19) satisfies both the differential equation and the stated boundary conditions.

4. Establish the claimed relation to the standard Feynman Tree Theorem.

The algebra used here resembles the causal core of FTT: a propagator is written as a retarded part plus a cut part, and closed retarded cycles vanish. However, the manuscript uses a finite-interval Dirichlet boundary-value Green function and a separable endpoint correction. Standard FTT starts with a Feynman propagator carrying an $i0$ prescription, replaces it by a retarded or advanced propagator plus an on-shell distribution, and integrates the cuts over on-shell phase space. The manuscript currently supplies neither that prescription nor a precise map from its endpoint term to an on-shell cut.

The authors should therefore do one of two things. They may derive the finite-interval identity from a genuine $0 + 1$ -dimensional FTT, specifying signature, boundary state, $i0$ prescription, cut normalization, and the role of the determinant phase. Alternatively, they should consistently describe the result as an FTT-inspired or FTT-like finite-interval loop-tree identity. Phrases such as “direct application,” “exact $(0 + 1)$ -dimensional reduction,” and the statement that the bulk-to-boundary factor is the analogue of an on-shell delta function are too strong without this map. The assertion following equation (10) that no phase-space measure appears because the boundary is zero-dimensional also requires justification.

5. Delineate the novelty relative to known determinant reductions.

The manuscript acknowledges Gesztesy and Makarov but only says that the strategy and physical interpretation are “fundamentally distinct.” That is not an adequate comparison. Their semi-separable-kernel analysis already relates Fredholm determinants of ODE Green functions to Volterra solutions, which is close to the analytic content of the present construction. Equation (25) is the familiar Forman/general Sturm–Liouville boundary determinant formula. The authors should compare hypotheses and formulas directly and state that the novelty lies, if established, in the graphical cut organization and its FTT interpretation. The literature on functional

determinants with arbitrary linear boundary conditions, including the work of Kirsten and McKane, should be discussed at the point where equation (25) is presented.

The bibliographic entry for Gesztesy and Makarov should also be checked: the cited article appeared in *Integral Equations and Operator Theory* **47** (2003), 457–497. The present entry gives different volume, year, and page data.

6. Revise the higher-dimensional and QFT claims.

The sentence that an IVP or Cauchy interpretation in higher dimensions is still lacking conflicts with the cited paper of Ossipov, which explicitly constructs discrete and continuum higher-dimensional matrix/operator initial-value problems. Likewise, saying that such a lattice formulation “presumably” exists overlooks a known block-recursion construction. The authors should explain whether their open problem is specifically an FTT interpretation rather than a higher-dimensional Gelfand–Yaglom formula.

More substantial obstacles arise in continuum QFT: $G_0 u$ is generally not trace class in $d \geq 2$, coincident Green functions require renormalization, boundary evolution becomes operator-valued, and its determinant needs a regulator and counterterms. Gauge theories add constraints, ghosts, and zero modes. These are not merely questions of interpreting otherwise straightforward trees. The statement in the introduction that extension of the derivation to QFT is straightforward also contradicts the later “Roadblocks” section.

To support the proposed physical significance, the authors should work out at least one regulated higher-dimensional example, for instance a scalar field on a finite spatial lattice or a separable background with explicit mode regularization, and show how the cut series, counterterms, and boundary evolution fit together. Without such an example, the QFT discussion should be shortened and clearly presented as an outlook rather than a result.

7. Add nontrivial checks of the general-boundary construction.

The general formula would be much more convincing with explicit boundary matrices. For Dirichlet conditions one may take

$$M_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

for which $\det \mathbb{M}_g = (U_g)_{12} = y_g(t_f)$, recovering the scalar formula. Neumann and periodic choices would test matrix ordering and the characteristic determinant beyond that special case.

A constant potential gives an especially simple exact check. With $u(t) = u_0$, $T = t_f - t_i$, and $q^2 = \omega^2 - gu_0$,

$$U_g(T) = \begin{pmatrix} \cos qT & \sin(qT)/q \\ -q \sin(qT) & \cos qT \end{pmatrix}.$$

The Dirichlet, Neumann, and periodic characteristic determinants are, respectively, $\sin(qT)/q$, $q \sin(qT)$, and $4 \sin^2(qT/2)$, up to boundary-matrix signs that cancel in the ratio. Including these checks, perhaps in supplemental material, would verify equations (19), (24), and (25) in cases with qualitatively different boundary data.

Minor comments

1. In equation (6), the numerator should be $\det \hat{\mathcal{O}}_g$, not the unindexed $\det \hat{\mathcal{O}}$.

2. The notation $y_g^{(n)}$ in equations (4)–(5) suggests that the coefficient itself still depends on g . A notation such as $y^{(n)}$ would be clearer.
3. Equations (5) and (21) should display a genuine n - or m -fold integration measure, or an explicitly ordered simplex. A single integral sign followed by $\prod_k dt_k$ is ambiguous.
4. The inner sum in equation (22) should explicitly say that it is over ordered compositions $m_1 + \cdots + m_k = n$, $m_j \geq 1$. These m_j are chain lengths, not the multiplicities m_i used in equation (9).
5. In the sentence preceding equation (14), $U(t_f, t_i)$ should be $U_g(t_f, t_i)$. The functional trace Tr and the finite boundary trace tr_∂ should be distinguished consistently throughout.
6. The introduction’s statement that cancellation of ultraviolet modes leaves sensitivity “only to low-energy spectral features” is too strong. All spectral shifts contribute; their high-energy asymptotics cancel sufficiently to make the relative determinant well defined in the one-dimensional setting.
7. Products of several cut chains are forests rather than single trees, and the Volterra terms are not literally classical paths. The graphical terminology and the meaning of each straight and wavy line should be defined in a short legend.
8. A paper centered on FTT should cite the original theorem and a modern treatment of its multiple-cut combinatorics. Related loop-tree identities with nontrivial boundary conditions should also be acknowledged.
9. The claims that recursive use of Forman’s theorem is impossible should be qualified. The pseudodifferential nature of the generic boundary operator obstructs a direct repetition of the same differential theorem, but special separable, product, and discrete block-recursion settings do admit further reduction.
10. The phrase “matrices have twice as many entries as lattice points” should refer to their linear dimension, not their number of entries.

Recommendation

I recommend **major revision**. The paper contains an elegant reorganization of the one-dimensional determinant identity, and explicit checks support the correctness of its formal algebra. Publication would be appropriate after the authors make the all-orders proof complete, correct the free matrix-kernel statements, state the determinant and boundary hypotheses, and sharply separate the known Forman/Volterra content from the new diagrammatic contribution. The title and strongest FTT/QFT claims should be retained only if the revised manuscript establishes the precise connection to the standard theorem; otherwise the language should be moderated. A regulated higher-dimensional example would substantially strengthen both the novelty and the relevance to the intended high-energy theory audience.