Referee Report

A discrete series gauge field at the late-time boundary of dS_4

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Recommendation: Major revision. The paper contains a useful and potentially publishable planar-patch realization of the photon discrete series, including explicit Bunch–Davies oscillator formulae and a transparent helicity split. The transverse lower-index mode and self-duality calculations are largely convincing. Two bulk equations must first be corrected: equation (2.6) is not the Gauss constraint in the stated raised-index notation, equation (2.7) omits a $6A^i/\eta^2$ term, and the late-time operators in equations (2.44)–(2.46) miss the already-fixed factor $\sqrt{\pi}/2$. The central representation-theoretic claim is also not yet established at journal standard: the proposed late-time norms are evaluated only as formal plane-wave densities, and the crucial map $\beta_i(\mathbf{k}) \mapsto k^{-1}\beta_i(\mathbf{k})$ is checked under dilatations but not shown to intertwine the full $\mathrm{SO}(4,1)$ action. These gaps affect the claim that both asymptotic operators furnish positive, unitary discrete-series modules.

Summary and overall assessment

The manuscript studies the free Maxwell field in the conformal planar patch of four-dimensional de Sitter space. In Section 2 the authors impose $A_\eta = 0$ and spatial transversality, normalize the two physical Bunch–Davies modes with the Klein–Gordon product, and canonically quantize the resulting transverse field. The exact half-integer Hankel modes make the late-time expansion particularly simple. The leading and subleading coefficients define transverse spin-one operators $\alpha_i(\mathbf{k})$ and $\beta_i(\mathbf{k})$, displayed explicitly in equations (2.46a) and (2.46b), with dimensions 1 and 2, respectively. Equations (2.60) confirm these dimensions from infinitesimal dilatations.

Section 3 relates the standard one-photon Hilbert space to a proposed late-time construction. The authors explain how infinitesimal de Sitter charges act on creation and annihilation operators. Dilatations and translations are worked out in equations (3.10) and (3.12); special conformal transformations are argued to preserve the physical content up to a null pure-gauge mode. The late-time states are then defined by acting with α_i or β_i on the Bunch–Davies vacuum.

Section 4 identifies the relevant exceptional-series labels with the photon module D_{01}, which in four spacetime dimensions is a direct sum of the two discrete series. For α_i, the proposed shadow intertwiner is

$$G'^{+}_{01}(\mathbf{k}) = k(\delta_{ij} - \widehat{k}_i\widehat{k}_j),$$

as in equation (4.17). It maps the dimension-one transverse operator to the dimension-two shadow $k\alpha_i$. Equations (4.23)–(4.33) use this operator to obtain a positive norm density. For β_i, the standard operator in equation (4.42) projects longitudinally and therefore annihilates the conserved transverse data. The manuscript instead defines $\beta'_i = k^{-1}\beta_i$ in equation (4.45), assigns it to the same quotient realization as α_i, and imports the preceding inner product through equation (4.47).

Sections 5 and 6 give the helicity decomposition. Because the Hodge star commutes with orientation-preserving de Sitter isometries, the self-dual and anti-self-dual Maxwell solution spaces are invariant. Equations (5.7)–(5.14) implement this statement in the Bunch–Davies basis. Equations (6.4) split each boundary operator into fixed-helicity pieces, and equations (6.7)–(6.11) relate those pieces to the two chiral field strengths. The authors then infer that single-particle states of opposite helicity do not mix under the de Sitter algebra. Section 7 summarizes the construction, records the late-time commutators, compares with earlier representation-theoretic treatments, and discusses a possible connection to conformal gauge fields in microscopic higher-spin models.

There is worthwhile content here. The formulas connecting the planar Bunch–Davies oscillators to the two boundary falloffs are concrete, and the use of self-duality to expose the two helicity modules is economical. The exact $H^{(1)}_{1/2}$ expansion supports the two powers, relative signs, and momentum dependence in equations (2.44)–(2.46), while the conformal invariance of four-dimensional Maxwell theory explains the flat-space form of the transverse lower-index mode equation. The overall normalization of those boundary operators is wrong, as discussed below. The split of the real field into mutually conjugate self-dual and anti-self-dual parts is physically sound. These points make the proposed result plausible after correction.

The current argument nevertheless moves too quickly at precisely the places where an invariant Hilbert-space structure must be established. Matching spin and dimension, or checking a single generator, does not by itself prove that two constrained function spaces carry equivalent $\mathrm{SO}(4,1)$ representations. Likewise, dividing an integrated delta-function norm by an infinite volume gives a useful density but does not define the positive completion of the module. Since unitarity of both late-time realizations is the principal claim of the paper, these are major rather than presentational issues.

Correctness and logical structure

The lower-index transverse mode calculation is internally consistent, but the preliminary constraint analysis is not. Writing $a(\eta) = 1/(H|\eta|)$, conformal invariance gives the Gauss law

$$\partial_i(A_i' - \partial_i A_\eta) = 0.$$

Equation (2.6), as written with $\partial^i = a^{-2}\delta^{ij}\partial_j$ and $A^i = a^{-2}\delta^{ij}A_j$, differentiates the factor a^{-2} in $\partial_\eta(\partial_i A^i)$ and is not equivalent to this constraint. In temporal gauge, Gauss's law states that $\partial_i A_i$ is time-independent; setting it to zero is an additional use of residual time-independent gauge freedom, subject to boundary conditions.

There is a second concrete error in equation (2.7). From the correct lower-index equation (2.8) and $A^i = H^2\eta^2\delta^{ij}A_j$, one obtains

$$(A^i)'' + \frac{4}{|\eta|}(A^i)' + \frac{6}{\eta^2}A^i - \boldsymbol{\partial}^2 A^i = 0 \qquad (\eta < 0),$$

so the displayed upper-index equation is missing the $6A^i/\eta^2$ term. The subsequent mode expansion is built from equation (2.8), so these errors need not invalidate its physical transverse sector, but the derivation and all claims of complete gauge fixing must be corrected.

After that correction, the polarization conventions in equations (2.14)–(2.20), the Klein–Gordon normalization in equation (2.29), and the transverse canonical projector in equation (2.31) fit together. There is, however, an additional normalization error when those modes are taken to late time. Expanding

$$H^{(1)}_{1/2}(k|\eta|) = -i\sqrt{\frac{2}{\pi k|\eta|}}\, e^{ik|\eta|}$$

and multiplying by the Klein–Gordon normalization $\mathcal{C} = \sqrt{\pi}/2$ from equation (2.29) produces coefficients $1/\sqrt{2}$, not $\sqrt{2/\pi}$. Thus equations (2.44), (2.46a), and (2.46b) all omit the factor $\sqrt{\pi}/2$. This is also visible internally: the correctly normalized field-strength modes in equation (5.3) agree with equations (6.8)–(6.9) only if the corrected α_i, β_i are used. With the manuscript's definitions, the right-hand sides of those equations require an extra $\sqrt{\pi}/2$. The positive norm densities in Sections 4.2 should accordingly be $1/2$, rather than $2/\pi$. The later conventionally rescaled operators in equations (4.33) and (4.49) are unaffected. The momentum-space dilatation laws in equation (2.60) then give $\Delta_\alpha = 1$ and $\Delta_\beta = 2$.

The geometric core of the helicity argument is also correct. On an oriented four-dimensional Lorentzian manifold an orientation-preserving isometry preserves both the metric and the volume form, and hence commutes with the Hodge star on two-forms. It follows that the two eigenspaces $\star F = \pm iF$ are invariant under the connected de Sitter group. The mode assignments in equations (5.7)–(5.11) are compatible with the stated polarization convention. This establishes invariant chiral solution spaces and strongly supports helicity preservation.

What is missing is the functional-analytic and group-theoretic bridge from these correct observations to the strongest conclusions. The authors do not write the full compensated special-conformal action on the transverse boundary representatives. The inner products are tested only on operator-valued momentum eigenstates, rather than defined for arbitrary wave packets. Most importantly, the argument following equation (4.45) uses the dilatation weight of $k^{-1}\beta_i$ as if it established the full intertwining relation. Special conformal transformations are the nontrivial test, especially because they do not preserve the chosen Coulomb representative without compensation. Until this step is supplied, the invariance of equation (4.47), and therefore the second realization of the unitary photon discrete series, remains an assertion.

Novelty and significance

The manuscript sits between several bodies of known work that it cites: the global-dS_4 identification of the photon one-particle space with discrete series, the function-space and shadow-transform description of exceptional representations, recent analyses of de Sitter photons, and standard conformal invariance of self-dual Maxwell sectors. The genuinely distinct contribution appears to be the combination of:

- explicit planar-patch Bunch–Davies oscillator expressions for both late-time falloffs;
- a proposed momentum-space norm for each falloff in the quotient realization; and
- an operator-level identification of the two helicity summands at late times.

This is useful for the de Sitter representation and cosmological-boundary community, but the novelty is presently diluted by a long review and by the incomplete proof of the second item. Moreover, $k^{-1}\beta_i(\mathbf{k})|0\rangle = -i\alpha_i(\mathbf{k})|0\rangle$, so the two proposed one-particle realizations are identical up to a phase on the states actually normalized. A revised version should explain what nontrivial content remains in treating both operator falloffs, state the new result as a precise module isomorphism, prove the full intertwining relations, and contrast the construction directly with the cited work of Sun, Rios Fukelman–Sempé–Silva, and Loparco–Penedones–Ulrich would be appropriate for a journal such as JHEP or PRD. Without that strengthening, the paper is mainly an explicit repackaging of known Maxwell and representation-theory facts.

Major comments

1. **Correct the Gauss constraint and upper-index field equation.**

   Equation (2.6) is not obtained from $\partial_\mu(\sqrt{-g}F^{\mu\eta}) = 0$ with the definitions given in its footnote. The correct conformal-coordinate constraint is $\partial_i(A'_i - \partial_i A_\eta) = 0$. Consequently, $A_\eta = 0$ does not itself demand $\partial_i A_i = 0$; it makes that divergence time-independent, after which a residual time-independent gauge transformation may be used to choose a transverse representative. The allowed gauge parameters and their spatial boundary conditions should be stated.

   Likewise, converting equation (2.8) to the contravariant component $A^i = H^2\eta^2 A_i$ yields a $6A^i/\eta^2$ term that is absent from equation (2.7). The authors should correct both equations and recheck the claimed covariant form (2.9)–(2.10), the canonical constraint discussion, and any later step that invokes complete gauge fixing. Since the explicit Bunch–Davies modes use the lower component A_i, the principal results may survive, but that should be demonstrated rather than assumed.

2. **Propagate the missing mode-normalization factor.**

   Equation (2.43) contains the already determined factor $\mathcal{C} = \sqrt{\pi}/2$. Combining it with the Hankel expansion (2.45) gives

   $$\alpha_j(\mathbf{k}) = \frac{i}{\sqrt{2}} \sum_{\lambda=\pm} \widehat{\epsilon}_j^{(\lambda)}(\mathbf{k})(a^\dagger_{-\mathbf{k},\lambda} - a_{\mathbf{k},\lambda})k^{-1/2},$$

   $$\beta_j(\mathbf{k}) = \frac{1}{\sqrt{2}} \sum_{\lambda=\pm} \widehat{\epsilon}_j^{(\lambda)}(\mathbf{k})(a^\dagger_{-\mathbf{k},\lambda} + a_{\mathbf{k},\lambda})k^{1/2}.$$

   Equations (2.44), (2.46a), and (2.46b) instead use $\sqrt{2/\pi}$, larger by $2/\sqrt{\pi}$. This should be corrected wherever the unrescaled late-time operators occur. In particular, the densitized values in equations (4.32) and (4.48) become 1/2, while equations (6.8)–(6.9) are consistent with the field-strength modes (5.3) only after this correction. The authors should rederive the Section 7 commutators from a single declared normalization convention. Although the arbitrarily rescaled "normalized" operators in equations (4.33) and (4.49) happen to retain their displayed form, the current manuscript is not internally normalized consistently.

3. **Define the full de Sitter action on the gauge-fixed boundary module.**

   The main text gives the infinitesimal bulk prescription in equation (3.7) and explicit momentum-space formulas only for dilatations and translations in equations (3.10) and (3.12). The special conformal generators are the essential missing case. They do not preserve $A_\eta = 0$, and the manuscript therefore invokes a divergence-free pure-gauge term in equation (3.8). The statement that this term is null holds only after discarding a spatial boundary contribution, while the states used later are plane waves and are not themselves of compact support.

   The authors should specify a dense wave-packet domain and its falloff conditions, construct the compensating gauge transformation, and give the resulting action of all ten generators on transverse $\alpha_i(\mathbf{k})$ and $\beta_i(\mathbf{k})$. At minimum, the special-conformal generator should be displayed and shown to preserve transversality modulo the stated quotient. The generators should close the $so(4,1)$ algebra and be anti-Hermitian under the proposed late-time forms. For special conformal transformations the authors should also verify that the Bunch–Davies complex structure is preserved, so that no positive/negative-frequency Bogoliubov mixing is hidden in the pure-gauge qualification. This would also resolve the present ambiguity between an infinitesimal algebra representation on a single planar patch and a representation that genuinely exponentiates to the connected group.

4. **Replace the formal norm density by an actual Hilbert-space construction.**

   Equations (4.23)–(4.33) evaluate $\langle \alpha_i(\mathbf{k}) | G\alpha_i(\mathbf{k})\rangle$, integrate it over all momentum, and divide by an infinite "volume of momentum eigenstates" Ω. The resulting $2/\pi$ is a positive norm density. It does not show that the individual delta-normalized states have finite norm, nor does it establish positivity and nondegeneracy for arbitrary linear combinations.

   A suitable revision would define, for transverse test functions,

$$|f;\alpha\rangle = \int \frac{d^3k}{(2\pi)^3}\, f^i(\mathbf{k})\alpha_i(\mathbf{k})|0\rangle$$

   and a sesquilinear form

$$(f,g)_\alpha = \int \frac{d^3k}{(2\pi)^3}\, f^i(\mathbf{k})^* \big[k\,\Pi_{\mathrm{T}}(\widehat{\mathbf{k}})\big]_{ij} g^j(\mathbf{k}),$$

   with the precise reality condition, quotient by longitudinal data, and completion stated. The authors should prove invariance, positivity, and nondegeneracy on this domain, and explain the treatment of $k = 0$, where the projectors and inverse powers of k used later are singular. The fixed-helicity restrictions of this form should also be written explicitly. The claims of "finite positive norm" should then be formulated for normalizable wave packets rather than momentum eigenstates.

5. **Prove the β_i isomorphism under the full conformal group.**

   This is the most important technical point. Equation (4.42) gives an operator proportional to $k^{-1}\Pi_{\mathrm{L}}$, so it annihilates the transverse current β_i. The manuscript then defines $\beta_i' = k^{-1}\beta_i$ in equation (4.45), checks its dilatation weight, and concludes that it belongs to the same representation space as α_i. Equal spin and scaling dimension do not suffice for this conclusion in a reducible exceptional module, particularly when invariant subspaces and gauge quotients are central.

   The paper should demonstrate explicitly that

$$I_{\beta\to\alpha} = k^{-1}\Pi_{\mathrm{T}}$$

   is the relevant nonzero, properly normalized intertwiner from the conserved current submodule D_{01} to $C_{01}'^{-}/F_{01}'$. In particular, it must satisfy $I\,T_\beta(g) = T_\alpha(g)\,I$, or the corresponding relation for every Lie-algebra generator, including special conformal transformations. Its kernel, image, inverse, domain, and $k = 0$ behavior should be specified. Once this is done, equation (4.47) should be derived as the pullback of the α form and its $\mathrm{SO}(4,1)$ invariance proved. As written, the central claim that $\beta_i|0\rangle$ gives a second unitary realization is not yet demonstrated.

   There is also a simple identity that the manuscript should confront. Equations (2.46a), (2.46b), and (4.45) imply directly on the one-particle space that

$$\beta_i'(\mathbf{k})|0\rangle = -i\,\alpha_i(\mathbf{k})|0\rangle.$$

Thus, the states whose norm is evaluated are the same momentum-helicity states up to a phase. This observation may provide the cleanest intertwining proof on the one-particle space, but it also means that "another realization" is not an independent Hilbert-space construction. The authors should state precisely what remains distinct—the full operator, its annihilation part, its scaling assignment, or its canonical pairing—and calibrate the novelty accordingly.

6. **Separate invariance, unitarity, and irreducibility in the helicity argument.**

   Equations (5.14) and (6.17) give a persuasive argument that the self-dual and anti-self-dual sectors are invariant and that the helicity label is preserved by the connected de Sitter algebra. Invariance alone does not prove that each summand is irreducible, and the positivity calculation in Section 4 is performed only after summing over both helicities. The manuscript should identify each chiral summand with D^+_{01} or D^-_{01} more explicitly—for example through its compact $SO(4)$ content, a standard classification theorem with all hypotheses stated, or an explicit isomorphism—and show that the proposed form restricts positively and nondegenerately to each summand.

   The relevant group component should also be specified. Proper orientation-preserving transformations preserve the Hodge eigenspaces, whereas a parity operation exchanges them. This distinction is precisely why the two chiral representations are mirror images and should be part of the representation statement rather than left implicit.

   The logic should also distinguish invariance of the complexified self-dual solution spaces from invariance of the Bunch–Davies positive-frequency subspace. A self-dual sector contains a positive-frequency mode of one helicity and the conjugate negative-frequency mode of the opposite helicity. Hodge-star invariance alone therefore does not establish equation (6.17); preservation of the Bunch–Davies complex structure is the additional input.

7. **Explain the real structure and the cross-helicity commutators.**

   For a real Maxwell field the self-dual and anti-self-dual parts are complex conjugate, not two independent real fields. Correspondingly, the commutators in Section 7 contain $\delta_{\lambda,-\lambda'}$: the $+$ and $-$ late-time operators pair nontrivially in equations (7.4)–(7.6). This is compatible with a direct sum of complex one-particle representations, but the manuscript should explain why it does not mean that the two operator algebras are independent sectors. It would be helpful to distinguish clearly among (i) the complexified classical solution space, (ii) the positive-frequency one-particle Hilbert space, and (iii) the Hermitian real field algebra. The role of the reality relations following equation (6.4) and of the symplectic pairing in Section 7 should then be transparent.

8. **Sharpen the literature comparison and the publishable claim.**

   The paragraph beginning after equation (7.6) is a useful start, but it should be reorganized around precise statements. Which formulas or module identifications are absent from the cited function-space treatment? Which part of the helicity split was already known globally, and which step is genuinely new in the planar patch? How does the quotient realization here compare with the recent de Sitter-photon analyses cited in the Introduction? A compact comparison, together with a theorem-like statement of the new planar-patch result and its assumptions, would make the significance much easier to assess. The speculative higher-spin discussion in Section 7.1 should remain clearly secondary to this well-defined contribution.

Minor comments

1. Equations (2.39)–(2.45) describe an asymptotic expansion rather than a literal limit containing two different powers of $|\eta|$. Replacing the limit notation by $\sim$ or by a stated asymptotic equality would avoid confusion.

2. The nonlocal inverse Laplacian in the transverse projector of equation (2.31) requires a prescription and a domain. This is tied to the $k = 0$ issue in the proposed quotient module.

3. The discussion after equation (5.7) reverses the abbreviations: SD stands for self-dual and ASD for anti-self-dual, not vice versa.

4. The proof around equations (5.13)–(5.14) would be cleaner and coordinate independent if it used $\mathcal{L}_\xi g = 0$ and $\mathcal{L}_\xi \epsilon = 0$, and hence $[\mathcal{L}_\xi, \star] = 0$, rather than emphasizing that a particular mixed component of the Levi–Civita tensor is constant in the chosen coordinates.

5. The statement preceding equation (6.15) that Lie derivatives commute with covariant derivatives is not true without qualification. For the identity actually needed here one may simply use Cartan's general relation $[\mathcal{L}_\xi, d] = 0$, so that $\mathcal{L}_\xi F = \mathcal{L}_\xi(dA) = d(\mathcal{L}_\xi A)$. Alternatively, the authors should state explicitly the isometry property that preserves the Levi–Civita connection.

6. The operator called "helicity" in equation (6.24) has eigenvalues $-i\lambda$ and is anti-Hermitian with the displayed convention. Either define the Hermitian helicity operator by multiplying it by i, or state explicitly that equation (6.24) uses the anti-Hermitian rotation generator convention.

7. After equation (6.17), a transformed momentum eigenstate is generally a superposition over output momentum $\mathbf{q}$. Equations (6.23)–(6.25) instead apply an $h(\mathbf{k})$ acting on the original external polarization. A physical helicity operator should act with the corresponding $\mathbf{q}$ on every output component. Thus equation (6.25) is not an independent proof of helicity preservation as written; that conclusion must rest on a fully justified version of equations (6.14)–(6.17).

8. The notation D_{01}, $D_{0\,1}$, and D_{0_1} is used interchangeably. It should be standardized, and the relation between this module and $D_{01}^{+} \oplus D_{01}^{-}$ should be stated once with unambiguous labels.

9. Since equations (7.4)–(7.6) are useful checks of the normalization, the authors should show one representative derivation and make clear whether the equations are distributions before or after smearing.

10. Statements that α_i is a boundary gauge field should say whether this means an equivalence class modulo gradients or the particular transverse representative chosen in Section 2. Special conformal transformations make this distinction operationally important.

11. The repeated contrast with AdS_4, according to which the $\Delta = 1$ Maxwell falloff is simply non-normalizable and ignored, should be qualified as a statement about the standard boundary condition. Alternative and mixed Maxwell boundary conditions, including boundary gauge-field quantizations, make the unqualified statement too broad.

12. The discussion immediately following equation (7.8) refers to states created by an undefined a_j; the intended operator appears to be α_j.

13. The action in equation (7.9) contains only the linear current coupling to $\mathcal{A}_j$. As written, it is not the full off-shell background-gauge-invariant scalar-QED action claimed in the next paragraph: $|(\partial + i\mathcal{A})\phi|^2$ also contains the $\mathcal{A}^2|\phi|^2$ seagull term and requires the scalar to transform. The authors should either include the full coupling (and the associated contact terms after integrating out the scalar) or state that only a linearized source coupling is intended.

14. The bibliography contains duplicate entries for the same work under "Sengor:2022kji" and "particles", and again under "Higuchi–Letsios" and "Higuchi:2025pbc". The citation to Sun should also be completed with standard publication or preprint information.

15. There are several small textual slips, including "discrete series series" in Section 7 and the reversed SD/ASD description noted above. A careful proofreading pass would improve an otherwise readable manuscript.

Recommendation

I recommend **major revision**. The explicit mode analysis and the self-dual helicity decomposition are convincing and potentially useful, and I do not see evidence that the basic physical picture is wrong. The paper's headline claim, however, is not merely that the two late-time falloffs have dimensions 1 and 2; it is that each supplies a positive unitary realization of the photon discrete series and that this realization splits into the two irreducible chiral modules. Establishing that statement first requires correcting the constraint, upper-index equation, and boundary normalization identified above. It then requires a full compensated $\mathrm{SO}(4,1)$ action, a wave-packet Hilbert-space definition, and a genuine intertwining proof for equations (4.45)–(4.47). If those points are supplied and the novelty is sharpened against the cited literature, the work should be suitable for publication.