Referee Report

"Monodromy, Hidden Conformal Symmetry and Soft Hair
in Kerr-MOG Black Hole"

Recommendation: Reject in the present form. The manuscript would require a ground-up reconstruction before it could be reconsidered. The thermodynamic first law used for the greybody-factor match is inconsistent with the mass-dependent MOG charge; the monodromy assignments are interchanged; the scalar conformal weights and several mass normalizations are wrong; and the covariant-charge calculation omits the MOG vector and scalar sectors. In addition, the principal thermodynamic results already appeared in an uncited paper by the same author, while the hidden-symmetry and absorption analysis closely specializes an uncited Kerr-Newman calculation. These are foundational correctness and originality problems rather than matters that can be repaired by local edits.

Summary of the manuscript

The paper considers the rotating Kerr-MOG metric, writes its outer- and inner-horizon thermodynamic quantities, and proposes a two-dimensional CFT description by combining several standard Kerr/CFT constructions. The claimed chain of evidence is as follows. First, the horizon monodromies of a neutral massless scalar are reorganized into left- and right-moving frequencies, from which temperatures T_L, T_R are read. Second, the low-frequency near-region radial equation is represented as an $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ Casimir, and its hypergeometric solution is used to rewrite the absorption probability in the universal finite-temperature CFT form. Third, conformal coordinates near the bifurcation surface are used to define two families of horizon diffeomorphisms. An Iyer-Wald charge supplemented by a counterterm is then claimed to yield $c_L = c_R = 12J$. Finally, Cardy's formula is used to reproduce the outer- and inner-horizon areas, and the product

$$S_+S_- = \pi^2 \left[4J^2 + \left(\frac{\alpha}{1+\alpha}\right)^2 (G_N M_\alpha)^4\right]$$

is interpreted as nonuniversal and nonquantized.

There is a useful algebraic core. Introduce

$$\mu = G_N M_\alpha, \qquad \beta = \frac{\alpha}{1+\alpha}, \qquad q_g^2 = \beta\,\mu^2.$$

Equations (3)–(4) imply

$$r_+ + r_- = 2\mu, \qquad r_+r_- = a^2 + q_g^2, \qquad r_\pm^2 + a^2 = 2\mu r_\pm - q_g^2, \qquad J = a\mu.$$

Thus the metric is algebraically the Kerr-Newman metric with the special constraint $q_g^2 = \beta\mu^2$. For $a > 0$, the square root occurring in equations (5), (22), (24), and (78) is simply a. It follows that equation (22) reduces to

$$T_R = \frac{r_+ - r_-}{4\pi a}, \qquad T_L = \frac{(2-\beta)\mu}{4\pi a},$$

and equations (24) and (78) reduce to $c = 12J$. With the paper's entropy normalization, the algebraic area identities in equations (79)–(83) also follow. These checks show why many displayed expressions look mutually compatible. They do not, however, establish a new MOG/CFT correspondence: they are primarily Kerr-Newman identities after a constrained-charge substitution.

Assessment of correctness, novelty, and significance

If a complete covariant-phase-space analysis of scalar-tensor-vector gravity showed that its matter contributions combine nontrivially to produce the same horizon Virasoro algebra, that could be a useful result. Likewise, a careful treatment of how a neutral probe changes a black hole whose gravitational charge is tied to its mass could clarify an interesting ensemble issue. Neither calculation is supplied. Instead, the metric-level Kerr-Newman formulas are carried over while precisely the MOG-sensitive terms are omitted.

The manuscript also describes the thermodynamic, monodromy, and soft-hair routes as three independent determinations of the same CFT data. That is not the logical structure of the calculation. The monodromy analysis can, after its convention problem is fixed, determine combinations interpreted as temperatures. Equation (24) is then selected so that equation (23) reproduces the already known area. Section 4 explicitly postulates the same central charge before equations (37)–(38). Sections 8–9 repeat the resulting algebraic substitution. Only the soft-hair calculation could independently derive a central extension, and in its current form it computes only the right-moving gravitational contribution, not the full MOG charge. The entropy agreements are therefore largely built in rather than independent predictions.

The originality problem is comparably serious. P. Pradhan, *Area (or Entropy) Products in Modified Gravity and Kerr-MOG/CFT Correspondence*, Eur. Phys. J. Plus **133**, 187 (2018), arXiv:1706.01184, is absent from the bibliography. That paper already contains the Kerr-MOG metric and horizon thermodynamics, the mass-dependent area product, the nonextremal temperatures now displayed in equation (22), $c_L = c_R = 12J$, and the extremal Cardy formulas repeated in the present conclusion. Moreover, Sections 4–5 of the present paper closely track Y.-Q. Wang and Y.-X. Liu, *Hidden Conformal Symmetry of the Kerr-Newman Black Hole*, JHEP **08** (2010) 087, arXiv:1004.4661. Under $M_{\mathrm{KN}} \mapsto \mu$ and $Q^2_{\mathrm{KN}} \mapsto \beta\mu^2$, their conformal coordinates, generators, Casimir, hypergeometric solutions, matching coefficient, first-law construction, CFT frequencies, and absorption formula map in sequence to equations (25)–(48) here. The organization and substantial portions of the exposition are also very close, yet this paper is not cited.

The existing Kerr-Newman literature is not merely analogous background: because the displayed metric has exactly Kerr-Newman form, it contains the metric-level calculation at issue. The manuscript must also confront B. Chen and J. Long, arXiv:1004.5039, on nonextremal Kerr-Newman hidden symmetry and scattering, and S. Haco, M. J. Perry, and A. Strominger, arXiv:1902.02247, on Kerr-Newman soft hair. The latter is cited, but the manuscript never identifies what changes when the Einstein-Maxwell gauge sector is replaced by the MOG vector/scalar system. As written, the paper does not isolate a result that is both new and established.

Major comments

1. **The theory, entropy, and phase space must be defined from one MOG action.**

   The paper introduces MOG as a scalar-tensor-vector theory, but equation (1) is subsequently treated as though it were a vacuum Einstein metric. No action, background vector field, scalar configuration, or approximation is specified. In the black-hole literature this metric is normally obtained in a truncation in which scalar fields are constant and the vector mass is neglected. The manuscript must state the precise truncation and verify that it is consistent for the perturbations and charge variations considered.

This omission directly affects the entropy and central charge. Despite retaining G_N, equations (5) and (81)–(82) use $S = A/4$, while equation (65) uses a gravitational prefactor $1/(16\pi)$. In a theory with a field-dependent or effective curvature coupling, the entropy is the Wald entropy of the full Lagrangian, and the presymplectic potential and Noether charge can receive scalar and vector contributions. The paper must either set $G_N = \hbar = 1$ consistently and define all masses and angular momenta in those units, or restore the coupling factors throughout. At present $c = 12J$ is not even dimensionally defined if $J = aG_N M_\alpha$ is kept as a geometric angular momentum while the coupling is explicit.

Most importantly, the soft-hair calculation in equations (64)–(78) uses only the pure gravitational Iyer-Wald expression. Kerr-MOG is not a vacuum Einstein solution. In the Kerr-Newman calculation of Haco, Perry, and Strominger, the electromagnetic surface charge and a compensating gauge parameter are included before the matter contribution is shown to vanish under the chosen horizon conditions. The corresponding calculation must be done for the MOG vector and scalar sectors. It is not sufficient to assume that their contributions vanish because the final metric resembles Kerr-Newman.

2. **The first law in equation (44) is incompatible with the mass-dependent gravitational charge.**

   This is the most consequential algebraic problem because equations (46)–(48) and the Frolov-Thorne rewriting in equations (59)–(60) depend on it. In Kerr-Newman notation the first law contains

$$d\mu = T_+\,dS_+ + \Omega_+\,dJ + \Phi_g\,dq_g, \qquad \Phi_g = \frac{q_g r_+}{r_+^2 + a^2}.$$

   At fixed α, the MOG constraint gives $dq_g = \sqrt{\beta}\,d\mu$. Directly differentiating the quantities in equation (5) therefore yields

$$T_+\,dS_+ + \Omega_+\,dJ = \left(1 - \frac{\beta\mu r_+}{r_+^2 + a^2}\right) d\mu,$$

   and analogously at the inner horizon. This is not equation (44) for $\alpha \neq 0$.

   There are two physically distinct choices, neither of which is addressed. If one varies within the fixed-α Kerr-MOG family, the vector charge changes with the mass and its work term must be retained. If the incident scalar is neutral and one holds q_g fixed, the final variation no longer obeys $q_g \propto M_\alpha$ at fixed α; an enlarged solution space or an α-variation must then be specified. A first law derived from the full MOG action is required before any left/right CFT energies can be defined.

   There is also a visible normalization failure. Substituting equation (47) and equation (22) gives $\omega_L/(2\pi T_L) = 2G_N^2 M_\alpha \omega$, whereas the first gamma function in equation (43) requires $2G_N M_\alpha \omega$. The denominator a/G_N in equations (46)–(47) introduces the extra factor. Thus equation (48) does not reproduce equation (43) with the displayed conventions even before the missing work term is considered.

3. **The monodromy assignments in equations (18)–(22) are interchanged.**

   Using the signed inner surface gravity of equation (5), one obtains

$$\alpha_+ = \frac{\omega(r_+^2 + a^2) - ma}{r_+ - r_-}, \qquad \alpha_- = -\frac{\omega(r_-^2 + a^2) - ma}{r_+ - r_-}.$$

   Consequently,

$$\alpha_+ + \alpha_- = 2\mu\omega, \qquad \alpha_+ - \alpha_- = \frac{2(2-\beta)\mu^2\omega - 2ma}{r_+ - r_-}.$$

But the phase identity in equation (20), with the coordinates in equation (21), requires the quantity multiplying t_L to be $2\mu\omega$. Equation (19) assigns that sum to ω_R, not ω_L. The labels are therefore reversed relative to equations (20)–(21). In addition, the time coefficient in equation (21) should be $-1/(2G_N M_\alpha) = -1/(2\mu)$ if the explicit units are retained.

The authors must distinguish carefully between a signed inner Hawking temperature and the positive local monodromy exponent convention, then derive every coefficient in equation (20). At present equation (22) is not derived from equations (18)–(21) as printed. The symbols $\omega_{L,R}$ are then reused in equation (47) for thermodynamic CFT energies with different normalization, which further obscures the inconsistency. Distinct notation should be used.

4. **The near-region and hidden-symmetry derivations contain specific mathematical errors.**

   The second inequality in equation (13), $\sqrt{\beta}\,\mu/r \ll 1$, does not follow from equation (12) or from the horizon bound. Near $r = r_+ \sim \mu$ it is generically of order unity. If imposed as an extra small-charge or far-overlap condition, it excludes the generic near-horizon regime used for the monodromy discussion. The near-region equation should instead be derived by a controlled power counting of the discarded $O(\omega^2)$ terms.

   The statement following equations (15)–(16) also assigns an irregular singularity at infinity to equation (14). The reduced equation is hypergeometric: infinity is a regular singular point. It is the unreduced radial equation (10), of confluent-Heun type, that has irregular infinity. This distinction matters in a paper whose central claim concerns monodromy.

   The displayed generators do not provide a reproducible correction. The symbol Υ_r in equations (32)–(33) is undefined, and $\bar{H}_0 = -2iM\partial_t$ in equation (33) retains the bare M where the preceding coordinate definitions require $G_N M_\alpha = \mu$. Equation (34) is written as an operator but already contains the Fourier labels ω, m inside $G_\pm$. The authors should display the differential operator in $\partial_t, \partial_\phi$, verify the Lie brackets and Casimir before acting on a Fourier mode, and explain explicitly that $\phi \sim \phi + 2\pi$ prevents these local generators from defining a global $SL(2,\mathbb{R})^2$ symmetry.

5. **The conformal weights and scalar solutions are inconsistent.**

   Equation (35) gives a Casimir eigenvalue $\ell(\ell+1)$. With the standard relation $h(h-1) = \ell(\ell+1)$, the physical root is

   $$h_L = h_R = \ell + 1,$$

   not (ℓ, ℓ) as stated in equation (36). Equations (42)–(43) themselves contain $\Gamma(1 + \ell + \cdots)$, so equation (48) can match them only with $h_L = h_R = \ell + 1$. The $\ell = 0$ mode makes the contradiction immediate.

   Equations (39)–(40) also require a fresh derivation. With

   $$z = \frac{r - r_+}{r - r_-},$$

   the outer horizon is at $z = 0$, and a manifestly ingoing solution is organized as $z^{-i\alpha_+}$ times a hypergeometric function of z. The displayed equations retain that prefactor but use $(r - r_-)/(r - r_+) = z^{-1}$ as the hypergeometric argument. This argument diverges at the horizon and, without a nontrivial analytic-continuation and branch prescription, generically produces both Frobenius behaviors rather than the asserted pure ingoing or outgoing mode. The matching coefficient in equation (42) is the one associated with the z-representation, reinforcing the inconsistency.

   Finally, equations (39), (40), and (42) contain $2M\omega$, while equation (43) silently replaces it by $2G_N M_\alpha \omega$. The same incomplete mass substitution appears in equation (33). The authors should substitute the corrected solution directly into equation (14), compute its Wronskian and horizon flux, and derive the matching coefficient and absorption probability with a single mass convention. Until then, the claim of a "precise" CFT greybody match is unsupported.

6. **The entropy-product argument uses the wrong sign criterion and does not establish quantization.**

   From equation (5), with the manuscript's signed $T_- < 0$,

   $$S_+T_+ = \frac{r_+ - r_-}{4}, \qquad S_-T_- = -\frac{r_+ - r_-}{4}.$$

   Therefore equation (6) is nonzero even at $\alpha = 0$, where the Kerr entropy product is mass independent. It cannot imply the conclusion drawn immediately after it. With this sign convention, the relevant identity is $S_+T_+ + S_-T_- = 0$. Indeed, if equation (44) were simultaneously true, that identity would imply mass independence at fixed J, contradicting equation (83). This exposes the missing charge-work term from another direction.

   Equation (83) is algebraically the standard Kerr-Newman product $\pi^2(4J^2 + q_g^4)$. It becomes mass dependent only after imposing the special relation $q_g^2 = \beta\mu^2$. The author's 2018 paper explicitly noted that this dependence may be an artifact of that postulate; the present claim that it is a fundamental distinction between MOG and Einstein gravity is therefore too strong. The paper must define which independent conserved charges are held fixed. Moreover, mass dependence of a classical formula does not prove that the product is "not quantized." Such a conclusion requires a quantum charge/angular-momentum spectrum and a stated quantization rule. It should be removed unless that analysis is supplied.

7. **The soft-hair calculation neither establishes both sectors nor the claimed charge algebra.**

   Equations (70)–(77) calculate a future-horizon, right-moving gravitational central term. There is no corresponding past-horizon or left-moving calculation. Equation (78) therefore establishes, at most, c_R within the assumed gravitational truncation; it does not derive $c_L = c_R$. The conclusion's statement that identical left- and right-moving charges were obtained is inaccurate.

   Several steps needed for a covariant phase-space result are also absent. No phase space or falloff conditions are specified; finiteness, integrability, conservation, and associativity are asserted rather than checked; and equation (66) is not shown to cancel the actual nonintegrable symplectic flux of the full theory. Equation (62) uses $\mathcal{L}_{\zeta_n} g$ where the variation labelled by m requires $\mathcal{L}_{\zeta_m} g$. Equation (65) uses $\bar{\zeta}$ although the subsequent calculation is described as the ζ_n sector. The contour or mode integral producing $\delta_{m+n,0}$ in equation (76), the normalization of the zero modes, and the possible linear-in-m cocycle shift are not explained. Appendix A then invokes a differently presented Wald-Zoupas term and says that the same central charge follows after a "simple integral," without reconciling the two constructions.

   The repeated square root in equation (78) equals $|a|$. The direct central term in equation (77), by contrast, is oriented and proportional to $J = a\mu$. Agreement therefore assumes $a, J > 0$. The same issue affects $\Omega_\pm$ in equation (5): a principal square root erases the sign of the spin. The orientation convention must be stated, or signs and absolute values must be restored consistently.

8. **The Cardy interpretation is assumed and partly circular.**

   Equation (24) is fixed by requiring equation (23) to reproduce equation (5); it is not derived from monodromy. Equations (37)–(38) and (79)–(82) then repeat that same identity. The paper should clearly distinguish an algebraic consistency check from an independent microscopic state count. The expression with $T_L - T_R$ for the inner horizon also needs a physical CFT interpretation; it is not automatically a second Cardy degeneracy formula.

The footnote in the conclusion states that $c_L T_L \gg 1$ is sufficient for Cardy's formula. That is not the general criterion. One needs the relevant modular-invariance, spectrum, ensemble, and high-temperature assumptions. In the extremal limit $T_L = (\alpha+2)/(4\pi\sqrt{1+\alpha})$ is of order unity, so large central charge alone does not turn the invocation into a derivation. It is acceptable in Kerr/CFT to state Cardy applicability as a conjectural assumption, but not to count the resulting equality repeatedly as independent evidence.

9. **The prior work must be cited, compared equation by equation, and the genuinely new result identified.**

   This is not a routine request to enlarge the bibliography. The same author's 2018 paper already contains the main thermodynamic and extremal CFT results, including formulas algebraically identical to equations (22), (24), and (83). Wang and Liu's 2010 Kerr-Newman paper maps essentially line-by-line onto equations (25)–(48). Some of the present inconsistencies are diagnostic of an incomplete specialization: equation (33) and equations (39)–(42) retain the old Kerr-Newman mass symbol, while equation (44) deletes the Kerr-Newman charge-work term even though the effective charge is now constrained by the mass.

   A revised work would need a transparent comparison table separating: (i) results already published for Kerr-MOG thermodynamics; (ii) results that follow immediately by substituting $q_g^2 = \beta\mu^2$ into Kerr-Newman; and (iii) results genuinely derived from the MOG action. At present category (iii) is empty. Given the close equation sequence and exposition, the originality and attribution issues should be resolved before technical reconsideration.

Minor and presentation comments

1. State the parameter domain, including whether $\alpha \geq 0$ and $a > 0$, and distinguish physical mass/angular momentum from geometrized parameters. Use M, M_α, and $G_N M_\alpha$ consistently.

2. Equation (1) should be described as a solution of the stated MOG field equations in a specified truncation, not simply as a solution of Einstein's equations. The appropriate primary Kerr-MOG black-hole reference should replace the Bullet Cluster citation currently used to support a mass definition.

3. The near-region calculation is for a generic nonextremal black hole at low frequency; Section 5's opening description as "near-extremal" is misleading unless an additional near-extremal limit is imposed.

4. Use a single symbol ℓ throughout. Correct the unmatched parenthesis after equation (14), define every abbreviation such as Υ_r, and repair the long equations whose present typesetting extends beyond the text block.

5. Equation (23) is a two-sector canonical Cardy expression and should not be called "chiral." Reserve that terminology for the extremal $T_R = 0$ limit.

6. The metric expansion in equations (50)–(51) is central to the soft-hair calculation but is only

asserted. State the order represented by the ellipsis, give enough intermediate inverse-metric data to reproduce the charge, and check all dimensions in B, D, E, U, V.

7. The introduction lists many broad Kerr/CFT references but omits the directly controlling Kerr-Newman and Kerr-MOG papers. The literature review should be shortened and refocused around the results actually used.

8. Claims such as "strong evidence," "precisely matches," and "independent methods" should be revised to reflect the assumptions and algebraic dependencies explained above. The statement that entropy products in Einstein gravity are universally quantized is too broad.

Conclusion

The manuscript contains a recognizable Kerr-Newman specialization, and a few of its algebraic identities are correct under the paper's implicit geometric-unit convention. The advertised advance, however, would have to reside in the MOG dynamics. Those dynamics are absent precisely where they matter: the first law, the entropy functional, and the covariant phase-space charge. Combined with the swapped monodromy basis, the inconsistent greybody derivation, the missing left-moving charge calculation, and the substantial uncited overlap with prior work, this prevents publication in a high-energy theory journal. A future submission would need to be a substantially new paper built from a specified MOG action and a corrected, fully attributed derivation.