

Referee Report

“Intersection Bounds for BPS Strings in Six-Dimensional Supergravity”

arXiv:2607.26197

Recommendation: Major revision. The lattice Zariski construction and the resulting coefficient bounds are potentially valuable contributions to the classification of six-dimensional supergravity. Publication in a journal such as JHEP or PRD should await a complete proof of the decomposition theorem, an explicit justification that the listed (-2) sectors are exhaustive, and either a proof or a substantial narrowing of the claimed finiteness result.

Summary and overall assessment

The manuscript studies mutual intersections of primitive BPS string charges in six-dimensional $\mathcal{N} = (1, 0)$ supergravity. Its central proposal is a decomposition

$$b_0 = P + N$$

of the gravitational anomaly vector into a part P pairing non-negatively with every primitive BPS generator and an effective part N whose support has negative-definite Gram matrix. The construction begins with a finite non-negative presentation of b_0 , minimizes $J \cdot N$ over a compact admissible set, and is presented as an intrinsic charge-lattice analogue of the Zariski decomposition. The manuscript further interprets P as an attractor direction and N as the tensionless SCFT sector selected by the corresponding flow.

The proposed decomposition is then paired with an external E-string charge e , for which $b_0 \cdot e = 1$. If

$$N = \sum_{i \in S} \alpha_i C_i,$$

equations (4.6)–(4.8) give $\alpha_i(C_i \cdot e) \leq 1$. For an isolated (-3) generator, the orthogonality equation gives $\alpha_{-3} = 1/3$, hence the advertised bound $C_{-3} \cdot e \leq 3$ in equation (4.13). For the three non-Higgsable clusters containing (-2) generators, the inverse intersection matrices yield the coefficients displayed after equation (4.20), with the weakest componentwise bound equal to seven. The M -matrix argument in equations (4.14)–(4.17) is intended to show that coupling these blocks to a larger negative sector can only increase their coefficients and therefore strengthen the bounds.

For an isolated gauged (-2) generator, whose Zariski coefficient can vanish, the paper instead uses the embedding of the induced gauge current algebra into the rank-one E-string $(E_8)_1$ algebra. The maximal Dynkin index of an $\mathfrak{sl}_2$ subalgebra of $\mathfrak{e}_8$, attained by the principal embedding, gives the much weaker bound $C_{-2} \cdot e \leq 1240$ in equation (4.24). The manuscript finally claims that these results establish finiteness of all primitive tensor intersection data up to charge-lattice duality.

The main idea is attractive. In particular, the elementary relation between the Zariski coefficient and an external unit pairing is useful, and the M -matrix monotonicity argument is well chosen. The numerical coefficients for the isolated (-3) , $(-3, -2)$, $(-3, -2, -2)$, and $(-2, -3, -2)$ blocks are internally consistent. I nevertheless do not think the central conclusions are yet established at journal standard. Some missing steps in Section 3 are repairable, but they are precisely the steps used throughout Section 4. More importantly, the universal (-2) statement rests on an asserted

exhaustion of possible support blocks, while the global finiteness statement explicitly leaves mutual E-string intersections to future work.

Logical structure and correctness

The physical inputs are stated in Section 2: an integral unimodular charge lattice of signature $(1, T)$, the classification of primitive tensionless string sectors, mutual non-negativity of distinct primitive generators in equation (2.8), and a finite populated non-negative presentation of b_0 in equation (2.9). These are strong and partly conjectural inputs, but they form a coherent conditional starting point if their precise status is kept visible.

The compactness argument following equations (3.2) and (3.3) does establish the existence of a minimizer for a fixed finite presentation. The subsequent idea for proving negative definiteness is also plausible: if a nonzero effective subcombination of the support paired non-negatively with all of its constituents, it could be shifted from N to P , lowering the objective. For a symmetric matrix with negative diagonal and non-negative off-diagonal entries, the resulting seminegativity criterion can indeed be converted into negative definiteness by a standard M -matrix argument.

However, Section 3 never proves the orthogonality asserted in equation (3.1). For every C_i appearing in N , one must show $P \cdot C_i = 0$. This follows by decreasing its coefficient by a sufficiently small $\epsilon > 0$: all inequalities against distinct generators improve, while the inequality against C_i remains valid for small enough ϵ if $P \cdot C_i > 0$. The objective then decreases because $J \cdot C_i > 0$. This short argument is indispensable. Without it, equation (3.4), the Pythagorean relation (3.5), the coefficient equations in Section 4, and the claimed uniqueness are not derived.

Once orthogonality is supplied, the isolated (-3) calculation is correct. The sign in the block equation and in equation (4.16) is also correct: with $M_{ij} = -C_i \cdot C_j$, one obtains

$$M_{BB}\alpha_B = q_B - M_{BR}\alpha_R,$$

and M_{BR} is entrywise non-positive. Thus a non-negative inverse gives the stated monotonicity. The three displayed cluster solutions are readily verified. What remains unproved is that these clusters, together with the separately constrained (-4) attachments, cover every (-2) generator that can occur in S under all of the stated assumptions.

The current-algebra observation for a gauged (-2) generator is reasonable at the level of Lie algebras. The value 1240 agrees with the principal $\mathfrak{sl}_2 \subset \mathfrak{e}_8$ Dynkin index in the normalization of equation (4.22). The manuscript should nevertheless derive carefully why the intersection multiplicity equals this affine level in the normalizations of equations (2.2), (2.3), and (4.1), and should distinguish a Lie-algebra embedding from a globally well-defined embedding of the six-dimensional gauge group.

Novelty, significance, and relation to prior work

The paper builds directly on the BPS-cone framework and finiteness arguments of Kim, Vafa, and Xu, as well as on the SCFT/LST classifications cited in Section 2. It addresses an explicit gap in the recent supergravity-block program of Hamada, Jeon, and Kim: small-gauge and gauge-free sectors can lead to intersection multiplicities not bounded by the mixed-anomaly equation. The comparison with the geometric F-theory bounds of Morrison and Taylor is therefore timely and useful.

The genuinely new component appears to be the use of an intrinsic Zariski decomposition of b_0 to convert $b_0 \cdot e = 1$ into a bound controlled by inverse negative-intersection matrices. This is conceptually simple but potentially significant, especially because the isolated (-3) bound does not invoke a current-algebra estimate. The M -matrix monotonicity also packages the extension from a local negative block efficiently.

At the same time, the mathematical ingredients behind existence, uniqueness, and inverse positivity are classical. The paper should distinguish the new physical application from standard Zariski and Stieltjes-matrix facts and cite a modern formulation of abstract Zariski decomposition if one is available. The attractor interpretation is more speculative than the lattice theorem: the cited black-hole and black-string literature does not by itself identify the generally real vectors P and N with physical integral charges or derive a six-dimensional flow ending at the proposed J_* .

If the gaps below are addressed, the work could become a useful contribution to the high-energy theory literature. In its present form, the broad claims of a purely EFT proof and of complete finiteness obscure the narrower result that is actually on firmest ground: conditional bounds on E-string pairings with specified negative blocks.

Major comments

1. The lattice Zariski decomposition must be stated and proved as a precise theorem.

Equation (3.1) includes effectiveness and negative definiteness of N , nefness of P against every primitive generator, and $P \cdot N = 0$. The minimization argument establishes only part of this as written. The authors should add the coefficient-variation proof of $P \cdot C_i = 0$ for every $i \in \text{Supp}(N)$, and then derive $P \cdot N = 0$ explicitly.

The negative-definiteness step should be given as a lemma with hypotheses and proof. The property obtained directly from minimization applies to subcombinations coefficientwise bounded by the selected N ; to invoke the linear-algebra criterion for every non-negative vector, one should note that an arbitrary vector can be rescaled into that box. The current footnote’s “without loss of generality” and maximization argument is too compressed for the main structural theorem.

Finally, the claim that the answer is independent of the initially chosen finite presentation needs a complete uniqueness proof. In the argument after equation (3.5), cancel the common generators of N and N' explicitly. The two remaining positive combinations lie in negative-definite supports, while all cross-pairings are non-negative; this is what makes their difference have strictly negative square unless it vanishes. The paper should also say what happens when $P = 0$ or $P^2 = 0$, and prove that a nonzero causal P has the future orientation required by equation (2.5). These details are not optional, because every coefficient bound in Section 4 uses the support orthogonality.

2. The attractor interpretation currently goes beyond the demonstrated lattice statement.

For $P^2 > 0$, equation (3.6) is a natural point in the closure of the tensor cone once its time orientation is established, and equation (3.7) follows for the support. This does not yet show that it is the endpoint of the physical attractor flow of a BPS black string. The coefficients in P and N are real; neither vector is shown to be an integral charge or the charge of a single physical string. The fact that a positive multiple of b_0 is assumed to be carried by a BPS string does not supply that missing identification.

The authors should either derive the constrained six-dimensional attractor equations, identify the charge whose flow is being followed, and discuss the null and zero cases, or consistently describe Section 3.2 as a convex-geometric analogy. The statement in Section 5 that the decomposition is the “physical content” of a gravitational BPS black-string attractor is not supported by the present derivation.

3. The universal bound in equation (4.21) requires an exhaustive classification argument that is presently absent.

The coefficient of a (-2) node can become arbitrarily small in a purely lattice-theoretic extension. For example, for a unit-linked chain

$$(-3, -2, \dots, -2)$$

with m consecutive (-2) nodes and $q = (1, 0, \dots, 0)$, solving $M\alpha = q$ gives coefficient $1/(2m+3)$ at the far end. The componentwise E-string bound there is therefore $2m+3$, not a uniform seven. The asserted maximum seven is valid only if the SCFT/LST classification really excludes every longer or more general support graph relevant to this argument.

The paragraph preceding equation (4.18) should consequently be replaced by a precise proposition, with references to the exact classification results, covering gauge enhancements, frozen sectors, multiple intersections, branches, and (-2) nodes attached to generators other than the three listed non-Higgsable clusters. It should also explain which cases are already bounded by anomaly or current-algebra constraints and why no remaining case can have a smaller Zariski coefficient. Without this exhaustion, the strongest new (-2) result is conditional on the displayed blocks rather than universal.

4. The isolated (-2) discussion must distinguish gauged strings, M-strings, and the precise current-algebra statement.

Equation (2.7) explicitly includes a gauge-free (-2) generator, the M-string, but the argument leading to equation (4.24) begins by assuming that C_{-2} supports a simple gauge algebra. Table 3 and several conclusions then shorten this to “isolated (-2) ”, which is misleading. The authors should state throughout that 1240 is a bound only for an isolated gauged (-2) sector, and explain separately how gauge-free M-string generators and their intersections are removed or identified by Weyl reflections.

The affine-level normalization also deserves a derivation from worldsheet anomaly inflow. In particular, explain why $k_{\mathfrak{g}} = e \cdot C_{-2}$ with no additional $\lambda_{\mathfrak{g}}$ or global-form factor, why the relevant currents must be realized as a subalgebra of the interacting $(E_8)_1$ sector, and which assumptions are needed when several gauge sectors meet the same E-string. The principal $\mathfrak{sl}_2$ has the largest Lie-algebra Dynkin index, but its exponentiated subgroup and compatibility with the six-dimensional matter global form are separate questions. Calling 1240 a conservative Lie-algebraic upper bound is defensible; calling it the complete physical classification is not.

5. The claimed finiteness of all tensor intersections is not proved in this manuscript.

Section 4.3 states that every primitive-generator intersection is fixed, bounded, or removed by charge-lattice duality. The next paragraph, however, explicitly says that intersections between distinct E-string generators have not been analyzed and are merely “expected” to be determined, with a proof deferred to another paper. Gauge-free M-string intersections and null critical-string sectors are likewise not treated systematically in Section 4. Moreover, bounded pairwise entries

alone do not establish finiteness unless the number and types of primitive generators and the relevant duality orbits are also controlled.

This issue directly affects the last sentence of the abstract, the final sentence of Section 4.3, and the main conclusion in Section 5. The cited supergravity-block work discusses Weyl reflections and infinite duality orbits, but the present paper neither defines the quotient being used nor states a theorem proving that it has finitely many intersection data. The authors should either import a precise, applicable theorem with all hypotheses and show that the new bounds close its final open cases, or remove the finiteness claim and present it as a future consequence contingent on the remaining E-string reconstruction. A result openly deferred to a companion paper cannot simultaneously be advertised as established here.

6. The status of the assumptions and the phrase “purely EFT” need substantial clarification.

The derivation is conditional on the completeness of the SCFT/LST classification, the assertion that every tensor-cone boundary carries a tensionless BPS string, and the effectiveness of b_0 , for which the paper invokes the strong Cobordism Conjecture. In addition, the exhaustive list of non-Higgsable clusters used in Section 4 has substantial F-theory origins. These may be well-motivated inputs, but they are not consequences of low-energy anomaly cancellation alone.

Section 3.3 further introduces a geometric argument using an elliptic Calabi–Yau fibration, the Type-IIB axio-dilaton, seven-brane deficit angles, and properties of isolated (-2) curves. This sits uneasily with the statements in the abstract and Section 5 that no string-theory or F-theory input enters at any point. Fortunately, an isolated (-2) node in $\text{Supp}(N)$ can be excluded directly once orthogonality is proved:

$$0 = b_0 \cdot C_{-2} = N \cdot C_{-2} = -2\alpha_{-2},$$

a contradiction to $\alpha_{-2} > 0$. The speculative geometric paragraph should be removed or clearly separated as a nonessential comparison. More generally, the paper should distinguish the unconditional lattice lemma from the physical theorem conditional on the classification and conjectural inputs.

7. The notation e and the scope of the bounds should be made consistent.

Section 2 uses $e^2 = -1$, $b_0 \cdot e = 1$ for the gauge-free E-string, but Table 1 also contains gauge-bearing (-1) generators with the same two numerical invariants. Equations (4.6)–(4.8) are lattice statements applicable to any external generator with unit b_0 pairing, whereas the $(E_8)_1$ argument applies specifically to a gauge-free E-string. The paper should define separate notation for a general (-1) generator and for the E-string, then state for each boxed bound whether it follows from Zariski positivity, anomaly cancellation, SCFT classification, or the E-string current algebra. This will also make the comparison with prior F-theory bounds more transparent.

Minor comments

1. Equation (2.3) uses a convention-dependent factor of two. Since this normalization later meets the worldsheet-level normalization, it would help to include the values of λ_i or point to the exact convention in the anomaly literature.

2. In equation (4.1), $c_L = 8$ should be identified as the interacting left-moving E-string sector available to the E_8 currents, rather than the unqualified full worldsheet central charge including center-of-mass degrees of freedom.
3. Table 1 should say whether it is intended to be exhaustive after low-rank representation coincidences, global anomalies, frozen sectors, and gauge-group global forms are imposed. If it is only a working list, label it accordingly.
4. The statement after equation (3.4) should read “future timelike or null” only after the time orientation has been proved. The zero vector should be treated separately.
5. The discussion of an “isolated (-3) ” in equations (4.9)–(4.13) means isolated within the support S , not necessarily disjoint from external generators. This distinction should be stated at its first use.
6. The component ordering in the coefficient table after equation (4.20) should be specified explicitly for each chain. This is especially useful for $(-3, -2, -2)$, where the two (-2) bounds differ.
7. The comparison in Table 2 calls the gap between seven and two “rather small.” A neutral quantitative statement would be preferable.
8. Section 5.3 contains an empty item before the final broad open problem. This produces a blank bullet and should be removed.
9. There are several correctable grammatical problems: “ b_0 necessarily” is missing a space in Section 3.3; “every other strings” should be “every other string”; “the only remaining case is a (-2) -generator is an element” should be rewritten; and “in any general cases” should be “in general.” The long F-theory paragraph in Section 3.3 also needs a complete grammatical and technical rewrite if retained.
10. The reference to the ongoing supergravity-block classification should identify the precise result used for the duality quotient and the version to which the present claims apply. Statements expected only in a future companion work should not be cited as established input.

Publication recommendation

The manuscript contains a promising organizing idea and several concise calculations that could be useful in future classifications of six-dimensional supergravity. The isolated (-3) bound and the blockwise M -matrix mechanism appear to be the strongest parts. At present, however, the proof of the decomposition is incomplete at a key step, the claimed universal (-2) bound lacks the necessary exhaustive classification argument, and the advertised finiteness theorem is contradicted by the paper’s own deferral of the E-string–E-string case.

I therefore recommend **major revision**. A revised manuscript would be suitable for reconsideration if it (i) supplies a rigorous decomposition theorem, (ii) proves or precisely conditions the (-2) exhaustion, (iii) treats the gauge-free and duality-quotient sectors explicitly, and (iv) aligns the abstract and conclusions with what is actually established.