Referee Report

Generalized Bergshoeff–de Roo identification in the Supergravity frame

Recommendation: Major revision. The proposed supergravity-frame implementation of the generalized Bergshoeff–de Roo identification is potentially useful. It packages a calculation that is very large in a duality-covariant frame into a compact generalized-curvature formula, and it may provide a practical route to higher orders. The present version, however, is not yet suitable for publication in JHEP or PRD. The normalization of the principal action is internally inconsistent: equations (8) and (112) imply coefficients that differ by a factor of -4 from the summary formula in equation (10), while the calculation in Section 5 follows the former normalization. A second mismatch occurs among equations (114)–(116) and (122) in the normalization of the Lorentz Chern–Simons form. In addition, the infinite-dimensional algebraic limit underlying the identification is treated by an unexplained trace prescription, the compact all-order equations are implicit recursive definitions whose formal solution is not established, and the paper leaves unresolved whether its three-form is equivalent to the conventional heterotic three-form already at $\mathcal{O}(\alpha'^2)$. These points concern the claimed result rather than presentation alone, but they appear addressable in a substantial revision.

Synopsis and overall assessment

The paper reformulates the generalized Bergshoeff–de Roo (gBdR) identification in variables intended to coincide directly with Lorentz-covariant supergravity fields. Section 2 reviews the extended $O(D, D+k)$ double-field-theory construction. The generalized frame $\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}}$ contains auxiliary gauge vectors, while the generalized flux $\mathcal{F}_{\underline{a}\overline{\mathcal{B}\mathcal{C}}}$ contains the composite gravitational connection. Their transformation laws match after converting an auxiliary gauge index into an antisymmetric extended Lorentz pair. Equations (5), (6), and (35) then motivate identifying the auxiliary gauge group with the extended Lorentz group and imposing $\mathcal{E}_{\underline{a}\overline{\mathcal{B}\mathcal{C}}} = \mathcal{F}_{\underline{a}\overline{\mathcal{B}\mathcal{C}}}$.

The main new ingredient is the alternative generalized-frame parametrization in equation (38), followed by the section choice and three gauge conditions in Section 3.2. These conditions reduce the extended Lorentz symmetry and express the generalized frame in terms of a proposed physical metric $\hat{g}$, two-form $\hat{b}$, dilaton $\hat{\phi}$, and the auxiliary objects that are eliminated recursively. Equations (48)–(58) show the transformations of the metric, gauge vector, two-form, and their curvatures before and after the identification. Equations (70) and (86) give the gauge-fixed frame, and equations (87)–(94) decompose the relevant generalized fluxes.

The paper defines a generalized torsionful curvature in equations (96)–(101). In this notation the proposed compact action is equation (8), or equivalently equation (98), with a term $-(4g^2 X_R)^{-1}\mathcal{R}^{(+)} \cdot \mathcal{R}^{(+)}$ and an implicitly defined $\hat{H} = d\hat{b} - \hat{\Omega}_{\rm CS}$. Section 4 expands the generalized curvature. Equations (103)–(105) give its components and the derivative operator used in the expansion; equation (107) supplies the prescription for an infinite extended trace; and equation (112) reduces the curvature contraction to one derivative-squared and two cubic-curvature structures at six-derivative order.

Section 5 checks only the pure-gravity sector. It argues that the candidate R^3 terms in equation (117) cancel, modulo integrations by parts and a metric field redefinition, leaving the square of a Lorentz Chern–Simons form. The section then contrasts the recursively defined $\hat{H}$ with the conventional three-form $\widetilde{H}$ in equation (121), displays an explicit higher-order contribution, and leaves their relation open.

The change of variables is an appealing organizational idea, and the compactness of equation (112) is striking. The relation to the authors' earlier gBdR construction, the independently derived α'^2 DFT action, and the recent mega-space/bootstrap approach is described clearly at a qualitative level. Nevertheless, compactness is meaningful only after the normalizations, the recursive construction, and equivalence to the known heterotic effective action are under control. At present the paper does not provide that control.

Logical structure and principal assumptions

The main conclusion rests on the following chain.

- The transformation match in equation (6) is promoted to an exact identification between auxiliary gauge and extended Lorentz data. Since no finite k solves the required dimension matching, the construction is understood through an infinite-dimensional limit.
- The two completeness relations in equation (35), together with the Killing metric, are assumed to remain valid in that limit. The divergent extended trace is replaced in equation (107) by X_R^{-1}.
- The three gauge choices in Section 3.2 are assumed to be mutually consistent with the section condition, the recursive identification, and the residual physical Lorentz symmetry to all derivative orders.
- The generalized flux equations are treated as a formal recursive system. Powers of g^{-2} generate the derivative expansion, while the square-root matrices $\chi^{\pm 1/2}$ and $\Box^{\pm 1/2}$ are expanded in the same bookkeeping.
- The generalized curvature defined through the auxiliary field strength is assumed to transform covariantly after the auxiliary indices are identified with composite Lorentz indices. Its square is then taken to generate the desired infinite tower.
- Agreement of the pure-gravity equivalence class at $\mathcal{O}(\alpha'^2)$ is used as the principal check on the construction, even though the dilaton, three-form, and mixed sectors are set aside and a field redefinition is invoked.

Several of these assumptions are standard ingredients of the gBdR program. What is needed here is a precise statement of their status in the new frame and a verification that they support the stronger claims made for the compact action.

Major comments

1. **The displayed action and its derivative expansion have incompatible overall coefficients.** Let
$$K = (g^2 X_R)^{-1} \mathcal{R}^{(+)\overline{\mathcal{C}\mathcal{D}}}_{mn} \mathcal{R}^{(+)mn}{}_{\overline{\mathcal{C}\mathcal{D}}}.$$
Equations (8) and (98) put $-K/4$ in the Lagrangian. Equation (112), on the other hand, states
$$K = \frac{b}{2} R^{(+)} \cdot R^{(+)} + b^2 \mathcal{I}_6 + \mathcal{O}(\alpha'^3),$$
where $\mathcal{I}_6$ is the bracketed six-derivative expression. These two equations therefore put
$$-\frac{b}{8} R^{(+)} \cdot R^{(+)} - \frac{b^2}{4} \mathcal{I}_6$$
in the action. This is also the normalization used at the start of equation (116). By contrast, equation (10), presented as the expansion of the full Lagrangian, contains $+bR^{(+)2}/2 + b^2\mathcal{I}_6$. Thus the headline formula differs in both sign and a factor of four from the derivation.

   The authors should first fix one master convention for b, X_R, the auxiliary field strength, and the curvature square, and then propagate it through equations (8), (10), (96), (98), (103)–(112), and

Section 5. A short table matching b to the usual heterotic α' convention would make the result checkable. Until this is corrected, neither the stated first-order normalization nor any coefficient at α'^2 can be assessed.

2. **The Chern–Simons normalization and the pure-gravity check are not mutually consistent.**

   With $\hat{b} = 0$, equations (57) and (114) give $\hat{H} = -\hat{\Omega}_{\text{CS}}$ and $\hat{\Omega}_{\text{CS}} = (2b/3)\Omega^{(+)} + \mathcal{O}(b^2)$. Combining this with equation (100) and the coefficient $-b/8$ used in equation (116), the cross term obtained from the displayed formulas is proportional to $b^2 R^{abcd}\nabla_a \Omega^{(+)}_{bcd}/6$ under the paper's standard unit-weight antisymmetrization. Equation (116) instead writes a coefficient $b^2/4$. Later, equation (122) starts the same expansion as $\hat{\Omega}_{\text{CS}} = b\hat{\Omega}^{(+)}_{\text{CS}} + cdots$, without explaining how that object is normalized relative to the form in equations (114) and (115).

   This may be a collision of two conventions, but it feeds directly into the only explicit validation of the six-derivative result. The revision should derive equations (114)–(116) line by line from equation (101), define every version of $\Omega^{(+)}$ once, and repeat the cancellation of equation (117) with the corrected coefficients. The equality should be stated at the level of the action density, including the common $\sqrt{-g}e^{-2\phi}$ factor.

3. **The infinite-dimensional identification and trace prescription require a mathematical definition.**

   The manuscript correctly notes after equation (35) that no finite k yields an isomorphism between the auxiliary algebra and $\overline{\mathcal{H}}$. The subsequent calculation nevertheless uses both finite-dimensional completeness relations in equation (35), traces over the extended vector space, and finally equation (107), $\delta_{\overline{\mathcal{E}}}{}^{\overline{\mathcal{E}}} - 2 = X_R^{-1}$, as a regularization. It is not specified which sequence of algebras and representations is being taken to infinity, how g and X_R scale, or why this prescription respects the local symmetry and cyclic identities used in deriving equation (112). Equation (111) also makes the physical parameter b depend on this limiting normalization.

   The authors should either provide a regulated finite-rank construction whose limit gives equations (35), (107), and (111), or state explicitly that the procedure is a formal replacement rule and prove that the coefficients through α'^2 are regulator independent. In particular, contractions should be performed at finite regulator before the limit, and the order of the $k \to \infty$ and derivative expansions should be specified. Referring to X_R simply as a Dynkin index is not enough because its normalization is doing essential regularization work.

4. **The compact all-order formula is an implicit recursion, not yet a demonstrated solution.**

   Equations (87)–(94), (97), and (101) define $\mathcal{F}$, $\mathcal{R}^{(+)}$, χ, and $\hat{H}$ in terms of quantities that contain the same objects again. This is a useful formal generating system, but the manuscript repeatedly calls the result exact or all order without establishing existence and uniqueness as a formal power series. The square-root matrices also require a choice of branch and an explicit order-counting rule. The half-integer orders displayed in equations (91) and (103) make it especially important to show why only integer powers survive in the physical action.

   A revision should formulate the recursion explicitly: identify the zeroth-order data, give the map that determines the coefficient at order n from lower orders, and show that no undetermined homogeneous component remains after the gauge conditions. Reproducing equations (103), (104), and (112) from that recursion in an appendix, or supplying a symbolic verification, would make the central calculation reproducible. The term "all order" should mean a formal series unless convergence or a nonperturbative definition is actually supplied.

5. **Lorentz covariance after gauge fixing needs a direct order-by-order check.**

The third gauge condition in Section 3.2 is accompanied only by the compensating relation $\Lambda_{\underline{ab}} = -\Lambda_{\overline{ab}} + \mathcal{O}(\alpha'^2)$. The paper then infers all-order covariance of the SUGRA variables and of the action. It should be shown that all three gauge conditions are preserved by the composite Lorentz transformation at the order actually used in equation (112), including the variations of $\chi^{\pm 1/2}$ and $\Box^{\pm 1/2}$. This is particularly relevant for equation (105): the operator uses different connections on the first and second curved Lorentz slots and, as the authors note, does not preserve the antisymmetry of $\mathcal{R}^{(+)}_{be}$. Its Lorentz transformation and the precise index ordering in equation (112) should therefore be demonstrated rather than asserted.

It would also help to separate three statements that are currently blended together: invariance under the original auxiliary gauge group, covariance under the extended double Lorentz group, and covariance under the residual physical Lorentz group after the field-dependent identification and compensating transformations.

6. **The relation to the known heterotic α'^2 action is central and cannot be deferred.**

   The pure-gravity calculation in Section 5 is too weak to establish that equation (8) represents the heterotic effective action in a SUGRA scheme. It deliberately discards $\nabla\phi$ and H terms, removes Ricci terms by a metric field redefinition, and does not compare the surviving mixed couplings with the amplitude- or symmetry-based actions cited in the bibliography. More seriously, equations (121)–(124) lead the authors to two possibilities: either the explicit $\hat{\Omega}^{(2)}$ is removed by identities and field redefinitions, or the construction defines a genuinely different Lorentz-invariant three-form. These possibilities have different physical implications, and the paper leaves them open.

   The revision should resolve this fork at least through $\mathcal{O}(\alpha'^2)$. An acceptable route would be to give the explicit local redefinitions mapping $(\hat{g}, \hat{b}, \hat{\phi})$ to a conventional heterotic scheme and show agreement of the full action, including H and dilaton couplings, modulo total derivatives and leading equations of motion. Alternatively, if no such map exists, the claims must be narrowed and the physical status of the new invariant completion must be tested. A comparison with the independently derived DFT action and the α'-bootstrap result should be quantitative, not left as future work.

7. **The partial check should distinguish identities from field redefinition equivalence.**

   Equations (118)–(120) combine Bianchi identities, integrations by parts, omission of non-pure-gravity terms, and removal of Ricci tensors by a metric redefinition. These operations do not show that the displayed local density contains no R^3 terms; they show, subject to the corrected coefficients, that its pure-gravity part belongs to an equivalence class with no on-shell independent R^3 invariant. The required metric redefinition should be written explicitly, and its induced dilaton and three-form terms should be retained when making a statement about the full action. This distinction is important because avoiding complicated field redefinitions is advertised as a principal benefit of the new frame.

Minor comments

1. Please specify at the outset whether D is kept arbitrary or fixed to ten. The introduction fixes $D = 10$, whereas most algebraic statements and the limiting construction are written for general D, p, q, k.

2. Equation (38) and the particular representative in equation (133) appear to disagree in the sign of the metric in the barred second column: the former contains $-(\hat{c} - \hat{g})\hat{e}/\sqrt{2}$, whereas the latter contains $-(\bar{c} + \bar{g})\tilde{e}/\sqrt{2}$. Please correct this or explain the intervening convention change.

3. Around equations (43)–(44), distinguish consistently between an ordinary inverse, a transpose inverse, and raising an index with $\hat{g}$ or κ. This is essential in a parametrization where the barred frame alone is not a vielbein for $\hat{g}$.

4. The source contains duplicate equation labels for the generalized flux and its transformation, and duplicate bibliography keys for the Hronek–Wulff and Gitsis–Hassler papers. These produce multiply-defined references. The isolated citation command immediately before the Jeon–Lee–Park bibliography item should also be removed.

5. Several indices in the formulas following equation (133) appear mistyped: barred gauge indices become underlined, and the expression for $\hat{e}_{\tilde{\mu}}{}^{\overline{\alpha}}$ contains an unrelated spacetime frame. Equation (149) and the preceding auxiliary formulas should be checked similarly for missing tildes and repeated punctuation.

6. Equation (121) should use $(db)_{mnp}$, not db_{mnp}. Please also use one notation consistently for $\hat{\Omega}_{\mathrm{CS}}$, $\Omega^{(+)}$, and $\hat{\Omega}^{(+)}_{\mathrm{CS}}$.

7. The statement that equation (112) has "three independent couplings" needs a specified invariant basis and equivalence relation. Independence can change after Bianchi identities, integrations by parts, and leading-order field redefinitions.

8. The manuscript should explain the half-integer bookkeeping in equations (91) and (103) and explicitly show its cancellation in physical quantities.

9. A careful copy edit is needed. There are corrupted accented names and en dashes, as well as typographical errors such as "generlaized," "in the our SUGRA scheme," inconsistent hats and bars, and mismatched indices. Several long displays also extend into the margins.

Publication recommendation

The SUGRA-frame parametrization could be a worthwhile contribution: it offers a conceptually direct set of variables and a potentially substantial computational simplification relative to the earlier DFT-frame expansion. The novelty is therefore credible, but its significance depends on showing that the compact expressions have the claimed normalization and belong to the heterotic effective-action equivalence class. I recommend major revision rather than rejection because the underlying strategy appears promising. A publishable revision should, at minimum, reconcile the coefficient conflicts, define the infinite-dimensional prescription and formal recursion, verify the residual Lorentz covariance, and settle the three-form comparison through $\mathcal{O}(\alpha'^2)$.