Referee Report

"Massive and massless particles in Mielke–Baekler geometries"

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Recommendation: Major revision

Summary and overall assessment

The manuscript studies massive and massless spinning worldline systems on the three-dimensional homogeneous geometries associated with the Mielke–Baekler (MB) algebra. It first relates the two algebra parameters p, q to the torsion and curvature of a canonical invariant connection, reviews central extensions and quadratic Casimirs, and constructs explicit homogeneous coordinates. Most of the particle analysis is then restricted to the teleparallel branch $p = 0$. In that branch the metric is either Minkowski or AdS_3, while the preferred invariant connection is flat and torsionful. The authors derive worldline actions by pairing Maurer–Cartan forms with coadjoint-orbit representatives.

For the massive particle, the manuscript finds regular and critical sectors separated by $m + qs = 0$. In the regular sector the boost variables are removed by second-class constraints and the metric trajectory is a timelike Levi–Civita geodesic, although the reduced symplectic structure remains spin-dependent. At the critical locus two additional gauge symmetries appear, the reduced phase-space dimension falls from four to two, and the authors rewrite the momentum and spin transport equations in a Mathisson–Papapetrou-like form. For the massless particle an analogous regular/critical analysis is presented, with nominal critical condition $\mu + qs = 0$. The final part classifies a class of point-canonical Noether transformations, argues that the spinless action admits the local three-dimensional conformal algebra, and rewrites that algebra in a basis adapted to the teleparallel MB translations.

There is substantial material of interest here. In particular, the side-by-side treatment of the coadjoint-orbit, teleparallel, Levi–Civita, and Dirac-constraint descriptions is potentially useful, and the massless analysis and its relation to conformal transformations appear to contain the clearest new results. Several of the central calculations are internally consistent: the invariant-connection construction in Section 2, the regular/critical rank changes in the massive system, the disappearance of the Levi–Civita force in the regular massive branch, and the change of conformal basis in equations (4.162)–(4.176) all fit together plausibly.

In its present form, however, I do not think the paper is ready for publication. There is an unanalysed rank-drop branch in the massless Hamiltonian system, an important gap in the derivation of the spinning Noether classification, and a flawed presentation of the regular-sector symplectic reduction in Appendix C. The normalization of the mass, spin, and the "spin tensor" used in the Papapetrou equation also requires a careful reconciliation with the standard AdS_3 basis and with the authors' earlier work. Finally, because the teleparallel $p = 0$ geometry is locally ordinary AdS_3 equipped with a different invariant connection and algebra basis, the paper must separate genuinely new physical statements from changes of variables or connection. These issues are repairable, but they affect the main claims rather than only the exposition.

Correctness and logical structure

The organization from invariant geometry to coadjoint orbits and then to Hamiltonian reduction is sensible. Equations (2.7)–(2.14) correctly explain how changing the reductive complement produces the MB brackets, and the flat condition in Section 2.5 is consistent with the two Cartan–Schouten connections on the AdS_3 group manifold. The explicit $p = 0$ coframe, metric, and curvature in equations (2.41)–(2.55) are also mutually consistent, modulo their stated coordinate-domain restrictions.

For the massive action, the two independent Euler–Lagrange equations (3.37) and (3.42) lead directly to the regular constraint $\epsilon^{A}{}_{BC}u^{B}E^{C} = 0$ and to the exceptional condition $m + qs = 0$. The determinant in equation (3.90) reproduces the same locus. The resulting count of one first-class plus four second-class constraints in the regular branch, and three first-class plus two second-class constraints in the critical branch, is plausible. The transposed-connection equation (3.50) and its Levi–Civita rewriting are an instructive part of the paper.

The massless analysis is less complete. Equation (4.42) itself shows that the constraint matrix degenerates at both $\mu = 0$ and $\mu + qs = 0$, whereas the text classifies the system solely by the second factor. Moreover, the claimed Noether theorem for nonzero spin rests on the unproved ansatz (4.75), and the formal inverse operator in equation (4.108) is not a solution of the remaining overdetermined system without an existence and integrability analysis. Appendix C then treats noncanonical reduced variables as though they had the canonical one-form $p_i\, dx^i$, omitting the Wess–Zumino contribution that is responsible for their nontrivial Dirac brackets. These points need to be corrected before the paper's conclusions can be assessed definitively.

Novelty and significance

The intended audience of JHEP or PRD should find the relation between MB frames, spinning probes, and constrained dynamics relevant. However, the novelty is currently difficult to judge because the manuscript only briefly mentions the earlier AdS_3 particle analyses in references [19] and [20]. The regular/critical sectors, geodesic motion away from criticality, and the additional critical gauge symmetry were already central results of reference [19], while reference [20] studied a considerably more general AdS_3 worldline action. Since equation (2.14) is related to the conventional AdS_3 algebra by an explicit invertible change of basis, the paper should give a precise dictionary to those works.

My reading is that the potentially publishable advance consists of the teleparallel interpretation of the transport equations, the systematic massless spinning construction, and the analysis of how the Wess–Zumino term restricts conformal lifts. If this is correct, those contributions should be identified sharply and proved at the same level of detail as the massive constraint analysis. It is especially important to distinguish an invariant observable effect of torsion from the appearance of a force after rewriting a flat torsionful connection in Levi–Civita variables.

Major comments

1. **The physical mass/spin dictionary and the normalization of the Papapetrou spin tensor must be resolved.**

   The basis change stated after equation (2.14) is essential for interpreting the results but is not carried through to the particle parameters. On the teleparallel branch define the conventional AdS translation generator by
   $$\overline{P}_A = P_A - \frac{q}{2} J_A.$$
   The corresponding curvature is $\Lambda = -q^2/4$, so the AdS radius is $\ell = 2/|q|$. Evaluating the coadjoint representative (3.3) on this generator gives
   $$\alpha(\overline{P}_A) = \left(m + \frac{qs}{2}\right) \eta_{A0}.$$
   Thus the mass parameter in the conventional AdS basis is naturally $M = m + qs/2$, while the Lorentz spin remains s. The manuscript's critical condition then becomes $M\ell = \pm s$, with the sign fixed by the chosen teleparallel branch, precisely the standard critical relation discussed in reference [19]. This dictionary should be stated explicitly in Section 3.2 and used when comparing with the literature. It would also clarify in what sense m, rather than M, is called the mass in the torsionful basis.

   There is a related apparent factor-of-two problem in equation (3.65), where the spin tensor is defined by
   $$S^{AB} = -2s\, \epsilon^{ABC} u_C.$$
   With the manuscript's conventions this gives $\frac{1}{2} S_{AB} S^{AB} = 4s^2$, so the tensor has spin magnitude $2|s|$, whereas the coadjoint Casimir identifies s as the spin parameter. The factor of two is what makes equations (3.71)–(3.73) take the standard $-\frac{1}{2} R \cdot S$ form, but its relation to the Lorentz moment map is not derived. The authors should check whether equation (3.65), an earlier coefficient, or the interpretation of s is misnormalised. At minimum they must derive the Noether Lorentz charge, state the spin supplementary condition (presumably $S^{AB} p_B = 0$), compute the invariant mass and spin, and show that equations (3.70)–(3.71) use the same physical normalization as equation (3.3) and references [19,20].

2. **Equation (4.42) contains an omitted massless branch.**

   The determinant is
   $$\det M = e^{4\varphi} \mu^2 (\mu + qs)^2.$$
   Consequently the statement following equation (4.42) that the regular sector is simply $\mu \neq -qs$ is false unless $\mu \neq 0$ has separately been imposed. At $p = 0$, the coadjoint-orbit rank condition in Section 4.1 also factorizes as $\mu(\mu + qs) = 0$, so this is not merely a Hamiltonian-coordinate artifact. In addition, the replacement of the three constraints Φ_A by (C, R, S) uses the nonzero coefficient μe^{φ} and ceases to be equivalent at $\mu = 0$. The critical basis in equations (4.51)–(4.52) likewise assumes $q\mu \neq 0$, as the text partially acknowledges.

   The cleanest resolution may be to define a propagating massless particle as having a nonzero null translational momentum and impose $\mu \neq 0$ explicitly in Section 4.1 and everywhere thereafter. If the pure-s, $\mu = 0$ coadjoint orbit is intended to be included, it requires a separate constraint analysis: the original action then has no spacetime kinetic term and does not describe the same massless particle system. In either case, the regular and critical hypotheses in every result of Sections 4.4–4.6 should be corrected.

3. **The central Noether-symmetry argument is not proved and its scope is overstated.**

   For $s \neq 0$, the conclusion that a Noether transformation must project to a Killing field depends on equation (4.75),
   $$\beta(x, u) = \chi_A(x) n^A(u).$$
   This is introduced with only the phrase "a little analysis". It is a strong exhaustiveness statement about solutions of the coupled differential equation (4.74), not a harmless parametrization. A complete derivation is needed, including the regularity or locality assumptions on β. Likewise, the integrability step from equations (4.80)–(4.82) to equation (4.83) should be shown rather than asserted, because it is what makes both Y and Z Killing fields.

   Necessity is also not sufficiency. Equations (4.98)–(4.110) leave an overdetermined first-order system for the internal lift. The expression $(1 + \lambda D_m)^{-1}$ in equation (4.108) is only formal: its domain, zero-modes, boundary data, and compatibility with equation (4.109) are not discussed. The six MB isometries should either be lifted explicitly to (β, γ), or their existence should be proved directly from the group action and reconciled with this PDE system.

   Finally, equation (4.57) deliberately restricts α^M to depend only on x. This is a reasonable class in which to test ordinary spacetime conformal transformations, but it is not the most general canonical Noether generator. The abstract and conclusions should therefore not claim an exhaustive classification of all Noether symmetries unless the restriction is removed. They may instead state precisely that proper spacetime conformal Killing vectors do not admit lifts within the specified point-canonical class for nonzero s. At the critical value $\mu = -qs$, equations (4.79) actually imply $Z = qY$, so equation (4.80) imposes the stronger chiral condition
   $$e_A Y_B = q\, \epsilon_{BCA} Y^C.$$
   The gauge-invariant residual is therefore a three-dimensional chiral Killing subalgebra, not an arbitrary element of the full six-dimensional AdS isometry algebra. This stronger conclusion should replace the less precise statement near the end of Section 4.6.

4. **The physical content of the critical Papapetrou equation and its gauge dependence need clarification.**

   The manuscript correctly notes after equation (3.73) that the critical momentum–velocity map is degenerate and that a spacetime trajectory is not fixed until the extra gauge freedom is chosen. Equation (3.101) indeed shows that the new first-class generators move x^M. It is therefore unclear in what sense the right-hand side of equation (3.71) is a "genuine" physical spin-curvature force, as claimed in Section 5, rather than a useful connection-dependent representation of gauge-redundant equations.

   The paper should identify the gauge-invariant observables in the critical sector and show how p_M, S^{AB}, and both sides of equation (3.71) transform under equations (3.100)–(3.101). It should then choose an admissible gauge and describe the resulting physical motion. This is also the place to reconcile the present formulation with reference [19], where the critical AdS_3 system was described in terms of AdS_2 geodesics. If the Papapetrou equation is equivalent to that result after the MB basis change and gauge fixing, demonstrating the equivalence would substantially strengthen the paper. If it is not equivalent, the origin of the difference must be explained.

5. **Appendix C does not correctly display the regular reduced symplectic structure.**

After imposing only $x^0 = \tau$, equation (C.2) writes

$$L_{\rm gf} = p_i \dot{x}^i + p_0.$$

This omits the term $s(\cosh v - 1)\dot{\varphi}$ from equation (3.74). That term is part of the symplectic potential and is precisely responsible for the spin-dependent reduced brackets in equation (3.95). Eliminating (v, φ) by second-class constraints does not turn the remaining (x^i, p_i) into canonical variables; Appendix C immediately exhibits their noncanonical Dirac brackets. Therefore equation (C.2), and the claim that the gauge-fixed kinetic one-form is simply $p_i\, dx^i$, cannot both be correct. The authors should derive the reduced symplectic potential or two-form consistently. The Hamiltonian $H_{\rm reg} = -p_0$ may still follow because the omitted term is first order, but it should be obtained without discarding the structure that produces equations (C.6)–(C.11).

There is also an index/sign mismatch between the main result (3.95), which contains the first of the following expressions, while equation (C.6) uses the second:

$$\epsilon^{ABC} u_C e_A{}^M e_B{}^N \qquad \text{and} \qquad \epsilon_{ABC} u^C e_A{}^M e_B{}^N.$$

The frame indices have not been raised correspondingly, so these expressions are not automatically identical in Lorentz signature. The same issue propagates to equation (C.7). A single consistent frame-index convention should be used and the sign checked.

6. **The relationship between the coadjoint orbit and the reduced phase space is described incorrectly.**

   The footnote following the critical phase-space count in Section 3.4 says that the coadjoint orbit and reduced phase space have the same dimension but are not the same space, because a trajectory is a point in the orbit and an integral curve in phase space. This conflicts with the Souriau construction used in Section 3.1. For these reparametrization-invariant systems, the coadjoint orbit is the space of motions, and the fully reduced phase space is normally symplectomorphic to that space (up to the covers and global issues already mentioned by the authors). Curves occur in the evolution or constraint space; Hamiltonian evolution on a gauge-fixed instantaneous phase space is a choice of description of the same reduced motions.

   The authors should formulate the relevant quotient maps explicitly and state whether their Dirac-reduced space is globally the orbit, only locally symplectomorphic to it, or a different cover. This would also provide a valuable check that the Kirillov–Kostant form agrees with the Dirac form in both regular and critical branches.

7. **The genuinely new content relative to known AdS_3 particle models must be isolated.**

   For $p = 0$, the spacetime metric is AdS_3, the algebra is $\mathfrak{so}(2,2)$ in an MB basis, and the two flat metric connections are the left- and right-invariant group-manifold connections. Hence the existence and dimension of the massive regular/critical sectors cannot by itself be presented as a new torsional effect. The manuscript should give an explicit field and parameter dictionary to references [19] and [20], state which equations are equivalent to known ones, and then identify the new statements. In particular, it should explain whether the transposed Weitzenbock description changes an observable or only reorganizes the same AdS dynamics.

   The massless action, its constraint reduction, and the proposed conformal classification appear more novel, but the literature discussion contains almost no comparison with earlier massless spinning/anyon or conformal worldline models. A focused comparison is necessary to establish significance. Once the proof and branch issues above are repaired, the authors should state clearly which massless results were not already implied by the standard coadjoint-orbit classification of $\mathrm{SO}(2,2)$.

Minor comments

1. In Section 2.3, the use of the $\mathfrak{h}$-invariant subcomplex to compute $H^2(\mathfrak{g})$ deserves a short theorem citation or a two-line complete-reducibility argument. Appendix A gives related background but does not directly justify the earlier central-extension calculation. Moreover, the inference from Lie-algebra cohomology to the absence of group central extensions should state the connectedness and global assumptions; vanishing Lie-algebra H^2 controls infinitesimal connected extensions, not every possible discrete global extension.

2. The manuscript should distinguish consistently among an affine autoparallel, a metric geodesic, and an unparametrized trajectory. In particular, Appendix B calls autoparallels of a general Weitzenbock connection "geodesics", which can obscure the later comparison with Levi–Civita geodesics.

3. The statement that the transpose of the MB Weitzenbock connection is the other Weitzenbock connection uses the fact that the torsion is a three-form, and hence that the transposed connection remains metric compatible. This assumption should be stated at that point rather than only becoming apparent in equations (3.54)–(3.56).

4. Equations (3.96)–(3.97) and the gauge choice in Appendix C are singular at $v = 0$. The text notes a coordinate restriction, but it should explain briefly how a second hyperboloid chart confirms that the constraint rank and gauge count are not artifacts of this parametrization.

5. In both massive and massless sections, it would help to display the Poisson algebra of the proposed first-class generators, at least weakly. The rank of the primary-constraint matrix identifies null directions, but a short closure calculation would make the gauge claims more transparent.

6. Section 2.6 says that the null hypersurface is not covered by the conformally flat chart. The radial variables ρ, k^M do fail there, but the actual coordinate map
$$y^M = \frac{2}{q\rho} \tanh\left(\frac{q\rho}{4}\right) x^M$$
has the smooth limit $y^M = x^M/2$ as $\rho \to 0$, and the conformal factor extends by its power series. The domain statement should be corrected. Global claims should still be distinguished from local ones on the AdS cover.

7. Equation (4.108) should not use inverse-operator notation without specifying what is inverted. Even if the general spinning lift is deferred, the equation can instead be presented as the differential equation it is.

8. The sentence in Appendix A that the simple algebra $\mathfrak{h}$ "acts reducibly" should read "acts completely reducibly". The distinction matters for the cohomological argument.

9. Please standardize the placement of Lorentz indices on ϵ_{ABC}, the coframe, and inverse frame throughout. The paper uses several visually similar forms whose equality depends on the Lorentzian sign from raising all three epsilon indices.

10. The conclusion should avoid saying that the Wess–Zumino term "produces" the regular/critical sectors without qualification. In the massive case the same exceptional AdS orbit structure is already visible in the conventional basis; the Wess–Zumino coefficient selects the orbit and controls the reduced symplectic form.

11. The normalization between equation (4.1) and the light-cone action (4.21) should be stated. With $P_\pm = (P_0 \pm P_1)/\sqrt{2}$ and the canonically dual basis used earlier, the moment in equation (4.1) pairs with the light-cone Maurer–Cartan form with an overall $\sqrt{2}$ (and sign, depending on which raised dual component is named π^-). Equations (4.21)–(4.22) appear to absorb this into a redefinition of μ, s, but no redefinition is announced.

12. The abstract should make clear that, although the geometric preliminaries treat general p, q, all explicit massive and massless particle dynamics are developed on the teleparallel branch $p = 0$.

13. In Appendix B, the second expression in equation (B.9) is missing its final $= 0$. Elsewhere, fix the unmatched parenthesis in the phrase "the parametrisation v, φ)" in Section 3.4 and the lowercase g^{MN} used near equation (C.10).

14. Section 4.4 uses a null vector ℓ normalized by $n \cdot \ell = -1$, whereas Appendix D introduces an auxiliary ℓ with $n \cdot \ell = +1$. These may be intentionally different vectors, but the notation should make that distinction explicit to prevent a sign ambiguity in the identity used in equation (4.76).

Recommendation

I recommend **major revision**. The manuscript contains a worthwhile geometric synthesis and potentially publishable new results for massless spinning particles, but the omitted $\mu = 0$ branch, the proof gap in Section 4.5, the Appendix C symplectic inconsistency, and the unresolved spin normalization affect central claims. A revision that corrects these points, gives a transparent dictionary to the standard AdS_3 particle parameters, and sharply separates invariant physics from the teleparallel choice of connection would merit renewed consideration.