Referee Report

Static topological dilatonic black hole with multiple horizons and its thermodynamics

Recommendation: Reject in the present form, with resubmission possible after a complete rederivation. The extension of the multi-horizon nonlinear-electrodynamics construction to Einstein–dilaton gravity could be useful, and a correct analysis of its canonical and grand canonical ensembles would be relevant to PRD or JHEP. The present version, however, has errors in the chain that defines the solution. Equation (5) does not follow from equation (2), and the coefficients and constraint in equations (19) and (21) do not set b_5 to zero. Consequently the finite electric field (22), and hence the metric (25) on which all subsequent thermodynamics is based, have not been shown to solve the displayed theory. There is also an independent algebraic error in the critical-charge formula: equation (84) does not solve equation (83). Finally, the advertised triple point is inferred from a plot rather than demonstrated by coexistence conditions; the parameters printed in Figure 10 give two nearby transition temperatures, not one three-phase coexistence temperature. These are central correctness issues, not matters of presentation.

Synopsis and overall assessment

The manuscript studies the action (1), in which a Liouville dilaton is coupled to a nonlinear electromagnetic Lagrangian represented by the infinite power series (2). The metric, electric, and scalar ansatzes are given in equations (6), (8), and (9). The integrated electric equation (13) is inverted as a power series. Equations (15)–(20) list the first inverse coefficients, while equation (21) is intended to tune all coefficients above α_4 so that the electric field terminates at equation (22). With a two-term Liouville potential, the author then obtains the metric function (25). Figures 1 and 2 exhibit examples with four or five positive roots.

Section 2.1 separates the electromagnetic and dilaton stress tensors. Equations (34) and (35) give the electromagnetic density and transverse pressure, and Figure 3 illustrates their signs for one five-horizon example. Equations (36)–(38) give the corresponding dilaton quantities. Section 3 derives the Hawking temperature (40), entropy (44), quasilocal mass (46), charge (49), and electrostatic potential (51), and checks the differential first law (53) with the nonlinear couplings and cosmological parameter held fixed.

Section 4 evaluates background-subtracted Euclidean actions. Equation (60) is identified with the grand potential at fixed boundary potential. The canonical action (76) includes the nonlinear electromagnetic boundary term and subtracts an extremal solution of the same charge. The paper then uses parametric plots of $W(T, \Phi_q)$ and $F(T, q)$ to discuss Hawking–Page-like behavior, small/medium/large black-hole phases, and a purported triple point in Figure 10. Equations (82)–(85) address canonical criticality, and equations (86)–(91) discuss local thermal and electric response.

Section 5 promotes the cosmological parameter to the pressure (92), derives the volume and equation of state, and obtains the usual mean-field critical exponents at a generic ordinary critical point. Equations (109) and (112) give Smarr formulas after treating $\alpha_2, \alpha_3, \alpha_4$ as thermodynamic couplings, and equations (113) and (114) state the corresponding extended first laws.

The paper contains a substantial amount of calculation and many potentially interesting phase structures. Its main conceptual ingredients are, however, already present in the multi-horizon construction of Ref. [53], the multicritical thermodynamics of Ref. [57], and the dilatonic black-hole literature cited throughout the manuscript. The new value would therefore have to come from a demonstrably correct dilatonic solution and a controlled ensemble analysis. Neither standard is met in the present version.

Logical structure and principal assumptions

The conclusions rely on the following chain of assumptions and results.

- The infinite series (2) is treated as a well-defined nonlinear electrodynamics, at least formally, and its electric equation is assumed to admit the inverse expansion (12)–(20).
- The higher α_j are tuned recursively so that the inverse series terminates after b_4. The finite expressions (22), (25), and all thermodynamic quantities are valid only on this constrained subspace of the infinite coupling space.
- A positive root used as r_+ is assumed to be the outermost regular event horizon. The family obtained by solving $U(r_+) = 0$ is then used as a parametric family of equilibrium saddles.
- The $\mu = Q = 0$ dilatonic geometry is assumed to be an admissible grand canonical reference, while an extremal solution of equal charge is used in the canonical ensemble.
- At fixed intensive data, the relevant real branch of equation (51) is assumed to be identifiable and the plotted free-energy curve is assumed to be the global thermodynamic envelope.
- Positivity of the two diagonal response functions in equation (87) is taken to characterize grand-canonical local stability, and ordinary cubic Landau behavior is assumed in the critical-exponent analysis.

Several of these steps are either algebraically false as printed or left without the checks needed to support the conclusions.

Major comments

1. **The truncation that defines the claimed solution is inconsistent.** Varying equation (2) with respect to A_μ gives a factor j in every term. That factor is absent from equation (5), although it reappears in the integrated equation (13). The manuscript must begin from one consistent field equation and rederive the solution from it.

   More seriously, formal reversion of equation (13) gives

   $$b_5 = 80\big(176\alpha_2^4 - 132\alpha_2^2\alpha_3 + 9\alpha_3^2 + 16\alpha_2\alpha_4 - \alpha_5\big)b^{-16\gamma}b_1^9.$$

   Equation (19) instead has $+132\alpha_2^2\alpha_3$. Even if that sign is corrected, $b_5 = 0$ requires

   $$\alpha_5 = 176\alpha_2^4 - 132\alpha_2^2\alpha_3 + 9\alpha_3^2 + 16\alpha_2\alpha_4,$$

   whereas equation (21) places an overall minus sign in front of this polynomial. Substitution of the printed equation (21) into the printed equation (19) plainly does not give zero. The stated value of α_6 is, in fact, consistent with the positive expression for α_5, which makes the contradiction especially clear.

   Because the metric (25) is obtained only after asserting the finite field (22), this error propagates to every result that follows. The author should provide the corrected recursion for all $\alpha_{j\geq 5}$, state whether equation (2) is a formal series or a convergent Lagrangian and on what domain, and substitute (22) and (25) explicitly into all three equations (3)–(5). A symbolic residual check, supplied in an appendix or ancillary file, would be appropriate. The Maxwell limit and the single-nonlinearity limit should be displayed as independent checks.

2. **Equations (84) and (85) do not follow from equation (83).** This can be seen without handling the general expression. Set $n = 3$, $\alpha = 0$, $b = 1$, and retain only α_2. Equation (83) becomes

$$r_c^{-2} - 6Q_c^2 r_c^{-4} - 56\alpha_2 Q_c^4 r_c^{-8} = 0.$$

Solving this quadratic in Q_c^2 gives

$$Q_c^2 = \frac{3r_c^4}{56\alpha_2}\left[-1 \pm \sqrt{1 + \frac{56\alpha_2}{9r_c^2}}\right].$$

By contrast, equation (84) reduces to an expression whose square root contains $(56\alpha_2/3)r_c^2$. It has the wrong power of r_c, the wrong numerical denominator, and, given the manuscript's statement that α_2 is dimensionful, the wrong dimensions. Its expansion therefore cannot yield equation (85). The literal g in the last exponent of equation (83) is an additional error.

The author should rederive equations (83)–(85), check them by direct substitution, and recompute all critical curves and claimed endpoints that use them. Since the paper's novelty rests in part on multicritical behavior, postponing the physical analysis of the multiple roots of equation (83) is not acceptable here.

3. **The triple point is not established, and the parameters in Figure 10 do not give an exact triple point.** Three-phase coexistence requires three distinct stable radii r_s, r_m, r_l satisfying

$$T(r_s) = T(r_m) = T(r_l), \qquad F(r_s) = F(r_m) = F(r_l),$$

together with the appropriate global-minimum and outer-horizon conditions. A graph that is acknowledged to be imprecise cannot replace these equations.

There is also a concrete discrepancy at the displayed parameter values. For $n = 3$, $\varepsilon = 1$, $\alpha = 0.1$, $b = 1$, $\Lambda = -0.15$, $(\alpha_2, \alpha_3, \alpha_4) = (-0.0893, 0.0295, -0.01453)$, and $q = Q = 0.82$, direct use of equations (40), (44), and (48) gives small–medium coexistence at approximately $T = 0.0440147$, with radii 0.45704 and 1.13109. At that temperature the large branch near $r_+ = 4.35343$ has free energy higher by about 1.06×10^{-2}. Medium–large coexistence occurs instead at approximately $T = 0.0442070$, with radii 1.13868 and 4.45200; there the small branch near $r_+ = 0.45708$ has free energy higher by about 6.48×10^{-4}. Subtraction of the common extremal mass does not change these differences. Thus the printed values describe two close first-order transitions, not a single triple point. The mismatch may be due to rounded parameters, but then the tuned parameters and numerical residuals must be reported.

A revised paper should solve the coexistence equations, tabulate all radii, temperatures, free energies, and residuals, and provide a phase diagram in an appropriate two-dimensional control plane. Figures 10–12 should distinguish genuine coexistence lines, critical endpoints, and merely metastable branches. Numerical data or code sufficient to reproduce every phase plot should be supplied.

4. **Grand-canonical branch selection and local stability are not under control.** The manuscript correctly notes that equation (51) can give several charges at fixed (r_+, Φ_q), but then proceeds without specifying which real roots are retained or comparing their grand potentials. A swallowtail or discontinuity in a selected parametric branch is not by itself a phase transition. At every (T, Φ_q) the author must enumerate all regular outer-horizon saddles, enforce the same boundary data, and take the global minimum of W. This is particularly important for the claimed zeroth-order transition in Figures 6 and 7.

Equation (87) is also not a complete grand-canonical stability criterion. The response matrix

$$\frac{\partial(S, q)}{\partial(T, \Phi_q)}$$

must be positive semidefinite. Besides a diagonal condition, this requires its determinant to be non-negative. Using standard Jacobian identities,

$$\det\left[\frac{\partial(S,q)}{\partial(T,\Phi_q)}\right] = \frac{C_q}{T}\,\epsilon_T.$$

Positivity of C_{Φ_q} and ϵ_T separately does not impose this condition. Indeed, equation (90) itself shows that C_{Φ_q} can be positive while C_q is negative. The stability assignments in Figures 14 and 15 must therefore be redone using the full Hessian.

The horizon branch requires equal care. A root with $U'(r_+) < 0$ is not automatically a naked singularity. For the AdS-like cases considered in the plots it generally signals that the selected root is not the outermost horizon, and a larger root may exist for the same geometry. Every plotted point should be checked against the complete positive-root set of equation (25), rather than classified only by the sign of equation (40).

5. **The Euclidean reference geometries and ensemble identities need a more careful derivation.** The grand-canonical background in equation (56) is obtained by setting $\mu = Q = 0$ and integrating from $r = 0$. The scalar (11) diverges there, and the geometry is not ordinary thermal AdS. The author should establish that this is an admissible saddle with the same induced metric and scalar boundary data, analyze its interior regularity and finite action, and explain the parameter range in which the subtraction is legitimate. The matching sentence below equation (56) should read $\beta\sqrt{U(R_b)} = \beta_0\sqrt{U_0(R_b)}$; as printed it contains β on both sides.

   The text also identifies the $\mu = Q = 0$ grand-canonical ground state with zero temperature. A horizonless thermal saddle instead admits arbitrary Euclidean period. Likewise, a black-hole saddle with positive W does not cease to exist; it is merely subdominant to the reference saddle. These distinctions matter for the Hawking–Page discussion.

   After correcting the underlying solution, the author should verify directly that equation (60) equals $\beta(M - TS - q\Phi_q)$ and that equation (76) equals $\beta(M - M_e - TS)$ for every retained branch. The dependence of the extremal reference on charge, pressure, and nonlinear couplings must be included consistently when derivatives are taken.

6. **The energy-condition claims exceed the calculation.** Equations (34) and (35) are parameter-dependent polynomials in a radial variable. Figure 3 proves, at most, that one chosen geometry has the stated signs in part of its static region. It does not prove the abstract's unqualified assertion that all electromagnetic energy conditions are fulfilled outside the black hole. The necessary and sufficient inequalities should be given at least for the truncated model, and their validity should be checked for every parameter set used in the phase analysis. The physical claim concerns the total stress tensor, so the electromagnetic and scalar contributions should also be combined.

   The discussion inside a horizon is not valid in the same orthonormal interpretation used outside. When $U < 0$, the coordinate directions t and r exchange causal character; the quantities called density and radial pressure in equations (36)–(38) cannot simply be continued with the same interpretation. Either formulate the interior statements with an appropriate timelike frame and invariant contractions or remove them. Finally, basic nonlinear-electrodynamics admissibility conditions, such as the signs governing the kinetic term and characteristic propagation, should be discussed alongside the energy conditions.

7. **The extended first law and Smarr relation are asserted on an ill-defined coupling space.** The truncation requires every $\alpha_{j\geq 5}$ to be a function of $\alpha_2, \alpha_3, \alpha_4$. Therefore the variations in equations (109), (112), (113), and (114) are variations along a constrained infinite-dimensional family, not ordinary partial derivatives of the action with all higher couplings fixed. This distinction must be made explicit, and the conjugates must be derived as total derivatives along that family.

The Euler argument is too abbreviated to resolve the issue. The manuscript should list the scaling weights, including $[\alpha_j] = L^{2(j-1)}$, state whether the dimensionful scalar parameter b is fixed boundary data or a variable, and account for the b dependence of Λ_1 in equation (24). If varying b changes the scalar boundary condition, an additional scalar work term may be required; if it is fixed, homogeneity of $M(S, P, q, \alpha_i)$ at fixed b must be demonstrated rather than assumed. The canonical Smarr relation must also include the induced variation of the extremal reference quantities.

8. **The novelty and scope should be sharpened after the corrections.** Reference [53] supplies the inverse-series multi-horizon construction, Ref. [57] already studies multicritical thermodynamics for the same type of nonlinear field, and Ref. [54] is described as giving a simpler Plebański formulation. The manuscript should identify precisely which result is new because of the dilaton and topology, and compare its phase boundaries quantitatively with the $\alpha \to 0$ and previously known cases. A collection of plots with chosen parameters is not sufficient to establish a general statement such as an arbitrary number of horizons or a robust new universality class. The allowed coupling domain, exact root counts, and persistence of the phenomena under parameter perturbations are needed.

Minor comments

1. Equation (49) omits α from the exponential $e^{-8\alpha\Phi j/(n-1)}$, in conflict with equations (2), (13), and (73).

2. The Q^2 denominator in equation (76) is printed as $n - 2 + a^2$ instead of $n - 2 + \alpha^2$. Equation (83) uses g instead of γ in its final exponent, and equation (98) has P/Pc instead of P/P_c.

3. Define explicitly $\Lambda \equiv \Lambda_0$ after equation (24), and state at the outset the restrictions on n and α, including every value excluded by denominators and the asymptotic analysis.

4. Figure captions alternate between (m, q) and (μ, Q) even though equations (25), (46), and (49) give nontrivial normalizations. Use one set of variables or state the conversion in every caption.

5. For Figures 1 and 2, list the numerical positive roots and verify their multiplicities. A plotted curve at finite resolution is not a reliable horizon count, especially close to extremality.

6. The phrase "negative temperature corresponds to a naked singularity" should be replaced by a root-by-root causal analysis. A negative signed surface gravity often identifies an inner horizon rather than a horizonless geometry.

7. The ordinary critical exponents derived from equation (100) require a nonzero cubic coefficient C. At a multicritical point that coefficient can vanish, so the conclusion should be restricted to generic ordinary critical points and separated from the triple-point discussion.

8. Correct the bibliographic and typographical errors, including the stray comma in the Ayón-Beato–García entry, "Nucl. Phys. D" in Ref. [21], "Phys. Rev." without the D in Ref. [44], and the many language errors in the prose. The manuscript requires a thorough copy-edit after the technical revision.

Publication assessment

The topic suits a high-energy theory journal, and a corrected solution with a quantitative phase diagram could be publishable. Here, however, the finite-field construction is inconsistent, a central critical formula is wrong, and the main multicritical claim fails at the quoted parameters. Repair requires a new derivation and phase analysis. I therefore cannot recommend publication in PRD or JHEP.