Referee Report

World-Line Actions in Weyl Geometry

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Summary of the manuscript

The manuscript studies the motion of a massive test particle in Weyl geometry from a gauge-theory perspective. After reviewing the standard square-root and einbein actions in Riemannian geometry, the authors formulate the Weyl autoparallel equation using the gauge-covariant derivative $\hat{\nabla}$. They emphasize the projective relation among the torsionless nonmetric connection, the metric connection with vectorial torsion, and the fully gauge-covariant description.

The main construction is developed in Section 3.3. In the integrable case, the authors introduce the compensator action

$$S = -\xi \int_\gamma \phi \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda,$$

and show that it produces the desired equation when $b_\mu = \partial_\mu \ln \phi$. For a purportedly general, non-integrable Weyl field they replace ϕ by the open Wilson-line functional

$$W(\lambda) = W(\lambda_0) \exp\left(\int_{\lambda_0}^{\lambda} b_\mu(x)\dot{x}^\mu \, d\lambda' \right).$$

Equation (52) then uses this W in the square-root action. The authors claim that the resulting functional is dimensionless, Weyl invariant, additive, and has the general Weyl autoparallel as its Euler–Lagrange equation. Equations (53)–(55) establish a concatenation identity along a fixed path.

Section 3.4 rewrites the construction after Stueckelberg breaking in terms of $B_\mu = b_\mu - \partial_\mu \ln \phi$ and argues that the usual Riemannian particle action is recovered when the dilaton is frozen and the massive Weyl field is decoupled. Section 3.5 studies affine parametrizations for an arbitrary assigned Weyl weight p of $\dot{x}^\mu$. It argues that inverse mass dimension, Weyl invariance, additivity, and affine parametrization cannot all be imposed on a proper-time variable in the unbroken phase. The section also gives an einbein action and, in equation (71), a local extended action in which W is constrained by a Lagrange multiplier.

These are worthwhile questions. A correct world-line variational principle for non-integrable Weyl geometry, and a precise separation between an invariant particle action and an operational clock time, would be relevant to the metric-affine gravity and Weyl-gauge communities. The integrable construction is also presented in a reasonably transparent way. However, the central extension to a non-integrable Weyl field omits the path variation of the Wilson line. This changes the equations of motion and invalidates the principal result in its present form.

Assessment of correctness, novelty, and significance

The logical structure of the paper is easy to follow: the Riemannian construction is used as a template, the integrable Weyl case is obtained with a local compensator, and a Wilson line is then proposed as its non-integrable replacement. The problem occurs precisely in the last step. A field that is covariantly constant in every spacetime direction cannot be replaced, inside a variational problem, by an object that is only covariantly constant along one selected curve. Varying that curve deforms the Wilson line and produces a field-strength contribution. Consequently, equation (52) does not give equation (31) for $F_{\mu\nu} \neq 0$.

This is not a small omitted detail. Equation (52) is identified as the central new action; the assertions about the general non-integrable theory, the einbein formulation, localization, and path-integral utility all depend on it. The remaining integrable result is much closer to familiar compensator and generalized-proper-time constructions. The novelty discussion must also be revised because the manuscript's own references [12]–[14], [16], and [29] substantially overlap with, or directly challenge, the proposed result.

The paper could become significant only if the authors either find a different variational principle that genuinely produces the desired non-integrable autoparallels, or change the stated result and analyze the actual nonlocal dynamics generated by equation (52). At present, the correctness and novelty standards of a journal such as JHEP or PRD are not met.

Recommendation

I recommend rejection in the present form. A substantially reconstructed manuscript could merit new consideration, but the required change goes beyond an ordinary revision: the main equation of motion, the claimed localization, the path-integral interpretation, and the novelty claim must all be reconsidered after the Wilson line is varied correctly.

Major comments

1. **The variation of the nonlocal action omits the Weyl field strength.**

   Let
   $$I(\lambda) = \int_{\lambda_0}^{\lambda} b_\mu(x(s))\dot{x}^\mu(s)\,ds, \qquad W(\lambda) = W(\lambda_0)e^{I(\lambda)}.$$
   For a deformation of the world line with fixed initial endpoint, integration by parts gives, up to the convention used for the ordering of indices on $F_{\mu\nu}$,
   $$\delta I(\lambda) = b_\rho(x(\lambda))\,\delta x^\rho(\lambda) + \int_{\lambda_0}^{\lambda} F_{\rho\mu}(x(s))\dot{x}^\mu(s)\delta x^\rho(s)\,ds.$$
   The first term is the endpoint term implicitly used in Section 3.3.2. The second term is the flux through the strip swept out by the deformed path and cannot be discarded in a world-line variation.

   Substitution into the variation of equation (52), followed by exchanging the order of integration, produces a term proportional to
   $$F_{\rho\mu}(x(s))\dot{x}^\mu(s)\int_{s}^{\lambda_f} W(\lambda)\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\,d\lambda.$$
   This is a history-dependent force. It is not generally proportional to $\dot{x}^\rho$ and therefore cannot be absorbed into a change of parametrization. It vanishes in the integrable case $F_{\mu\nu} = 0$, but it is present in precisely the non-integrable case for which equation (52) is introduced.

   The authors should perform the complete functional variation of equation (52), including all endpoint and path-deformation terms. If the expression above is confirmed, the statement following equation (48) and the conclusion following equation (55) must be withdrawn. The paper must then either be restricted to integrable Weyl geometry or reformulated around the actual nonlocal equation of motion.

2. **Equations (43) and (45) are not equivalent for a general Weyl background.**

   Equation (43) asks for a nonzero object satisfying $\hat{\nabla}_\mu W = 0$ in all spacetime directions. For a scalar of Weyl charge -1, its integrability condition is
   $$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]W = -F_{\mu\nu}W = 0.$$

Thus, wherever W is nonzero, equation (43) requires $F_{\mu\nu} = 0$. Equation (45), by contrast, is only the pullback condition $\dot{x}^\mu \hat{\nabla}_\mu W = 0$ along one chosen path. It always has a one-dimensional Wilson-line solution, but it does not supply a spacetime solution of equation (43).

This distinction also explains the failed variation in the previous comment. A path-dependent W is not an ordinary scalar function of the endpoint unless a prescription for a whole family of paths is given; its derivative under path deformation contains curvature. The authors must remove the assertion that equation (45) is "simply a rewriting" of equation (43), state precisely what space of histories W is defined on, and redo every step that uses the stronger spacetime equation.

3. **The local constrained action (71) is not shown to restore the claimed autoparallel dynamics.**

   Introducing W and a multiplier η can provide a local description in an enlarged world-line state space, but it does not make the field-strength term disappear. Varying

   $$\eta(\dot{W} - b_\mu \dot{x}^\mu W)$$

   with respect to x^μ produces a term proportional to $\eta W F_{\mu\nu} \dot{x}^\nu$, together with terms involving the multiplier equation. The equation obtained by varying W generically sources η, so setting this contribution to zero is not justified. Eliminating the constrained variables carefully should reproduce the nonlocal tail of equation (52), rather than the pure autoparallel asserted in the text.

   The manuscript should display all Euler–Lagrange equations following from equation (71), including those for x^μ, W, η, and the einbein. The variational boundary conditions for the first-order (W, η) sector must be specified. The Weyl weight, engineering dimension, and reparametrization law of η must also be given. Classical equivalence cannot be claimed until these equations and boundary terms are matched explicitly.

4. **Pathwise concatenation is not yet a path-integral composition law.**

   Equations (53)–(55) establish an algebraic identity only when the value $W(\lambda_b)$ at the join is transported from the preceding segment. Consequently, the action of the segment from b to c is not a functional of that subpath and its spacetime endpoint alone; it retains a state variable that records the earlier path. A standard kernel $K(x_c, x_b)$ therefore does not follow from the displayed additivity.

   If the intended quantum description uses equation (71), the natural state space is enlarged from x^μ to (x^μ, W), and kernel composition would require a matching condition and integration measure for W at each join. The role of the charged, dimension-one datum $W(\lambda_0)$ must likewise be explained: how is it fixed across varied histories, what physical field or boundary state supplies it, and does this choice introduce a scale? These issues must be settled before statements about path-integral quantization are made.

5. **The paper must confront the literature that already identifies the same variational obstruction.**

   Reference [16], Heisenberg's *Autoparallels and the Inverse Problem of the Calculus of Variations*, states in Section III.1 that the generalized nonlocal proper-time functional fails to reproduce generic autoparallels and reduces to a local action when the relevant one-form is exact. This is in direct tension with the sentence after equation (55) that the present setup readily satisfies the conditions of that work.

   Even more directly, Appendix B, equations (B-5)–(B-7), of the manuscript's reference [29] derives the field-strength term in the total variation of a Wilson line. That result is exactly the term omitted in Section 3.3.2. References [12]–[14] also already discuss the nonlocal generalized proper-time functional, its covariance, and its clock interpretation.

The authors should compare equation (52) term by term with these earlier functionals and explain what remains novel after the variation is corrected. The potentially new elements appear to be a gauge-theory interpretation and an auxiliary-field packaging, rather than the Wilson-line functional itself. The manuscript also needs a careful comparison between the Helmholtz conditions in reference [16] and the Finsler-metrizability analysis in reference [17], rather than treating them as automatic support.

6. **The affine-parametrization and proper-time argument contains an algebraic mismatch and proves only an ansatz-dependent statement.**

   Immediately before equation (61), the manuscript states correctly that $-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ has Weyl charge $2(p+1)$. Its gauge-covariant logarithmic derivative must therefore contain $2(p+1)b_\mu$, not $(2p+1)b_\mu$ as printed in equation (61). Equation (62), which contains $W^{2(p+1)}$, confirms the former coefficient. Equations (61) and (62) are not equivalent as written, and equations (63)–(65) should be rechecked after this correction.

   There is a related dimensional issue in the quadratic formulation. With the mass dimensions used in the manuscript, equation (66) requires $[\mathcal{E}]_M = M^{p-q}$. Hence $W^q\mathcal{E}$ has mass dimension p, whereas equation (69) sets it equal to the dimensionless quantity $1/\xi^2$. This is consistent without further explanation only for $p = 0$, or after a change of world-line parameter whose dimensions must be stated.

   More conceptually, equations (63)–(65) examine the family built from the chosen W and the assigned weight p. They do not by themselves exclude all relational clock observables, additional matter reference fields, or other invariant functionals. The authors should formulate a precise proposition with explicit assumptions on admissible fields, locality, endpoint data, engineering dimensions, and reparametrizations. Otherwise the conclusion should be narrowed to the statement that no member of the specific W^{p+1} family simultaneously has the four desired properties.

7. **The broken-phase formulas and physical interpretation require correction.**

   From equation (56),

   $$B_\mu = b_\mu - \partial_\mu \ln\phi, \qquad \text{so} \qquad b_\mu = B_\mu + \partial_\mu \ln\phi.$$

   Combining this with equation (48) gives

   $$W(\lambda) = \frac{W(\lambda_0)}{\phi(\lambda_0)}\phi(\lambda)\exp\left(+\int_{\lambda_0}^{\lambda} B(\lambda')\,d\lambda'\right).$$

   The minus sign in equation (57) is therefore incorrect. Equations (58) and (59), which define $\Omega = +\int B$ and use $e^{+\Omega}$, already employ the opposite sign and are internally inconsistent with equation (57).

   In addition, spontaneous symmetry breaking makes B_μ massive; it does not set B_μ identically to zero. The Riemannian action is obtained only after the further low-energy/background truncation $B_\mu \to 0$ and the constant-ϕ limit. Any residual exchange or background value of the massive field produces corrections that must be derived before their range, suppression scale, or relation to a second-clock effect can be asserted. The abstract and conclusions should distinguish the broken phase from this additional Riemannian limit.

8. **The physical test-particle model and the status of free fall need to be specified.**

Equation (30) is postulated as the appropriate free-fall law. In a metric-affine or Weyl theory, whether a physical wave packet follows metric geodesics, connection autoparallels, or a scalar-force trajectory depends on the matter action and its coupling to the independent fields. The action (35) describes a particle with a position-dependent mass $\xi\phi$ and is consistent with the stated integrable equation, but the manuscript should explain which matter excitation this point particle represents and whether the coupling is universal.

This is particularly important because the discussion later says that, in the minimal Standard Model scenario, only the Higgs couples directly to the dilaton, whereas the manuscript often speaks of an arbitrary massive particle. The relation among the world-line coupling ξ, masses induced indirectly through the Higgs sector, the equivalence principle, and the claimed clock phenomenology should be made explicit. The definition $\phi^2 = -R$ in equation (22) also requires a real, nonzero branch and a domain on which division by ϕ is valid; alternatively, if ϕ is an independent auxiliary field used to linearize R^2, that status should be stated.

Minor comments

1. Substituting the gauge choice (18) into the GR einbein action (14) gives

$$S = \frac{1}{2}\int \Big(m^2 g_{\mu\nu} x'^{\mu} x'^{\nu} - 1\Big)\, d\tilde{\tau},$$

   not equation (19) as printed. Factoring m^2 outside the whole parenthesis makes the constant term incorrectly normalized and combines terms of different mass dimension.

2. Eliminating the einbein from equations (66)–(67) recovers equation (44), or equation (52) after substituting the Wilson-line solution. It does not recover the integrable action (35) unless the additional restriction $W = \phi$ is imposed.

3. The reparametrization formula in the footnote before equation (8) appears to be missing Jacobian factors. For $\tilde{\tau} = \tilde{\tau}(\lambda)$, the transformed non-affinity is $f_{\tilde{\tau}} = f_\lambda/\dot{\tilde{\tau}} - \ddot{\tilde{\tau}}/\dot{\tilde{\tau}}^{\,2}$.

4. Equation (21), $u^\mu = \dot{x}^\mu$, chooses unit norm and corresponds to equation (9) only after fixing the normalization constant. The text should not call it equivalent to the arbitrary-k condition without this qualification.

5. The connection introduced in equation (6) generally has torsion. If "projectively equivalent" is intended in the conventional torsion-free sense, the transformation should be symmetrized; otherwise the broader convention being used should be stated.

6. The sentence opening Section 3 says that the GR action gives equation (10) in an arbitrary parametrization. Equation (10) is the affine form; the arbitrary-parametrization equation is (5).

7. The square-root/einbein equivalence assumes nonzero W and a nonzero coupling ξ; the displayed positive solution in equation (67) also requires a sign convention for ξ or the use of $|\xi|$. The $\xi = 0$ massless limit is a separate constrained system rather than the limit of the displayed elimination.

8. The lower endpoint λ_0 is at one point allowed to be $-\infty$. The convergence of the Wilson line, the gauge transformations permitted at the endpoint, and possible global or holonomy dependence should be specified.

9. The label “charges” is used for both equations (29) and (34), which produces a multiply defined cross-reference. References [27] and [29] are duplicate entries for the same preprint, and one of those entries has an extra closing bracket.

10. The manuscript would benefit from a careful copy edit. Examples include “lenghts,” “a interesting subject,” the stray brace in “the} time-like curve,” and several subject–verb and punctuation errors. The use of `\today` in the abstract should also be removed. The long review of standard GR material could be shortened in favor of a complete derivation of the central nonlocal variation.

11. Broad introductory statements that Weyl quadratic gravity is a renormalizable, anomaly-free, viable quantum theory should be tied to a specific action, spectrum, and quantization prescription. The acknowledged ghost issue and the assumptions behind the anomaly statement are too important for an unqualified characterization.

End of report