Referee Report

Quantum Fisher information of the Klein–Gordon, $\boldsymbol{\phi^4}$, and Dirac vacua

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Summary

The manuscript studies the quantum Fisher information (QFI), equivalently four times the pure-state fidelity susceptibility, of several vacuum families with respect to a mass parameter. The authors use the Euclidean half-space correlator in equation (26) and apply it to a free Klein–Gordon field, a scalar theory with a quartic interaction through first order in λ, and a free Dirac field in several spacetime dimensions. The principal claims are that the scalar QFI density scales as m^{d-2}, that the quartic interaction produces a divergence in $3+1$ dimensions and reduces the oscillator QFI in $0+1$ dimensions, and that the Dirac QFI is finite in $1+1$ dimensions but ultraviolet divergent in $2+1$ and $3+1$ dimensions. The paper also proposes a holographic interpretation of the massless $2+1$-dimensional scalar result.

The general topic is worthwhile. A consistent comparison of scalar and fermionic vacuum fidelity susceptibilities, especially with a controlled interacting calculation, could be useful at the interface of quantum information and quantum field theory. The manuscript is also commendably explicit about many intermediate integrals. Unfortunately, several central coefficients do not survive elementary independent checks. More seriously, the loop formula used for the interacting theory omits a gamma-function factor; this invalidates both advertised interacting results and eliminates the basis for the main physical conclusion. The treatment of ultraviolet renormalization, the spinor trace, and the holographic claim also requires substantial reconstruction.

Recommendation

Major revision. The present version is not suitable for publication in a journal such as PRD or JHEP. A publishable revision would require a complete recalculation in a single, explicitly stated regularization and renormalization scheme, together with independent canonical checks of the free theories and the $0+1$-dimensional interacting limit. Because these changes affect the paper's principal quantitative results and interpretation, they go beyond local corrections. If the corrected interacting calculation produces a well-defined universal result and the novelty is positioned accurately, a substantially revised manuscript could merit reconsideration.

Major comments

1. **The regulated quantity and the Euclidean conventions must be defined consistently.**

   Equation (26) integrates a connected correlator of local deformation operators on opposite sides of a sharp Euclidean-time interface. The regularized construction with the two insertions separated from the interface by ϵ is displayed and then commented out immediately before equation (26). Taking $\epsilon \to 0$ is not innocuous for composite operators. It is precisely where the contact and short-distance singularities encountered later in the Dirac and interacting calculations arise. The paper should begin from a finite spatial volume and a common ultraviolet regulator, retain the interface regulator until renormalization is complete, and state which quantity—the total QFI, its density, or a renormalized finite part—is being compared across theories. A radial position-space cutoff in one section and dimensional analytic continuation in another do not define a common observable.

There is also a basic sign inconsistency. Equations (20)–(25) use the weight $\exp(-S_E)$, whereas the scalar Euclidean Lagrangians in equations (28) and (43) have negative kinetic, mass, and quartic terms. Taken literally, the Gaussian and ϕ^4 functional integrals are unbounded. The propagator and the perturbative minus sign in equation (45) instead correspond to the usual positive Euclidean scalar action. The intended convention should be fixed throughout. The Euclidean Dirac operator in equations (75)–(77) and the propagator in equation (81) should likewise be checked by explicitly showing that their product is the identity distribution, including the overall sign.

2. **The $d = 1$ reduction in equation (36) is incorrect.**

   The angular parametrization in Appendix A degenerates when there is only one spatial relative coordinate. For a translation-invariant function one may perform the time integrations first and write

   $$\int_{\tau<0} \mathrm{d}\tau \int_{\tau'>0} \mathrm{d}\tau' \int \mathrm{d}^d x \, \mathrm{d}^d x' \, F(R) = V_d \int_0^\infty \mathrm{d}t \, t \int_{\mathbb{R}^d} \mathrm{d}^d r \, F\left(\sqrt{t^2 + r^2}\right).$$

   For $d = 1$, the remaining domain is a half-plane and its angular factor is 2, not 2π. The formula in equation (36) assigns the sole angle both the range $[0, \pi/2]$ and the azimuthal range $[0, 2\pi)$, producing an extra factor of π.

   This error is visible from a canonical mode calculation. The scalar vacuum is a product of oscillators of frequency $\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2}$, and therefore

   $$\frac{I_m^{(KG)}}{V_d} = \frac{m^2}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(\boldsymbol{k}^2 + m^2)^2}.$$

   In $d = 1$ this gives $I_{m,1+1}^{(KG)} = V_1/(8m)$, not $\pi V_1/(8m)$ as stated in equation (40). The $d = 1$ Dirac coefficient is affected by the same angular error. The authors should treat $d = 1$ separately in Appendix A and use canonical mode decompositions to verify every free-field coefficient.

3. **The loop identity in equation (63) is missing an essential factor, invalidating the ϕ^4 results.**

   The tadpole branch already contains internal inconsistencies. The factor m^d present in equation (50) disappears in equation (52), making that formula dimensionally incorrect. Moreover, equations (48), (50), and (52) carry an overall minus sign for positive $\lambda\Delta(0)$, but the $d = 1$ and $d = 3$ entries in equation (53) change it to a plus sign. Restoring the missing power of m, the preceding equations give $C_{1,3+1} = -\lambda V_3 \Delta(0)/(64\pi m)$, not the positive term used in equation (68). This sign problem is also a warning that a divergent first-order bare correction cannot be interpreted before mass renormalization.

   For $D = d + 1$, the standard Feynman-parameter integral contains

   $$\frac{\Gamma\left(a + b - \frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma(a)\Gamma(b)} \int_0^1 \mathrm{d}x \, \frac{x^{b-1}(1-x)^{a-1}}{[m^2 + P^2 x(1-x)]^{a+b-D/2}}.$$

Equation (63) replaces the numerator by $\Gamma(a+b)$ and hence omits $\Gamma((3-d)/2)$ when $a=b=1$. In $d=3$ the omitted factor is $\Gamma(0)$: the four-dimensional bubble is logarithmically divergent. After regularization it also has a nontrivial finite dependence on P^2. Thus equation (66), $L_{3+1}(P)=1/(16\pi^2)$, is not a valid loop integral. It cannot be converted into the contact term used in equation (67), and the claim $C_{2,3+1}=0$ does not follow. Consequently equation (68), including the conclusion that the displayed tadpole alone characterizes the interaction-induced divergence, is not established.

Even if one temporarily assumed the incorrect constant in equation (66), squaring $L_{3+1}=1/(16\pi^2)$ would produce a factor π^{-4}, whereas equation (67) contains only π^{-2}; the factor of λ also disappears in its second equality. Thus the displayed contact-term normalization does not follow even from the manuscript's own premise.

The same omission gives a decisive finite-dimensional check. Direct evaluation of the loop defined in equation (61) for $d=0$ gives

$$L_{0+1}(P)=\frac{1}{m(P^2+4m^2)}.$$

Equation (70) instead simplifies to $2/[\sqrt{\pi}\,m(P^2+4m^2)]$. With the correct loop, $C_{2,0+1}=-3\lambda/(32m^5)$, rather than $-3\lambda/(8\pi m^5)$. Standard perturbation theory for

$$H=\frac{p^2}{2}+\frac{m^2q^2}{2}+\frac{\lambda q^4}{24}$$

then yields

$$I_m=\frac{1}{2m^2}-\frac{11\lambda}{32m^5}+O(\lambda^2),$$

not equation (74). This oscillator result should be used as a mandatory normalization check before the field-theory calculation is reconsidered.

4. **The interacting QFI requires renormalized parameters and renormalized composite operators.**

   Even after correcting equation (63), equations (45)–(68) are expressions in bare correlators. The insertion conjugate to the mass is the composite operator ϕ^2, which renormalizes and can mix with lower-dimensional operators and interface counterterms. The coincident propagator $\Delta(0)$ in equations (45) and (46) is not by itself a physical, interaction-generated infinity of metrological information. In particular, the statement following equation (46) that it diverges only for odd d reflects poles of dimensional regularization; a cutoff also exposes power divergences in dimensions where dimensional regularization assigns a finite analytic value.

   The authors must specify whether the estimated parameter is a bare mass, a renormalized mass at scale μ, or a pole mass, and include the corresponding mass and composite-operator counterterms in Figure 1 and in the correlator. They should separate regulator-dependent power terms, logarithmic running, and any scheme-independent finite or universal contribution. Only after that calculation can one decide whether the interaction raises or lowers a physically defined QFI density. The present inference of a zero QCRLB from an unrenormalized tadpole is not meaningful.

5. **The spinor trace is lost in the Dirac calculation, and equation (85) does not follow from equation (84).**

Equation (79) correctly writes a trace over spinor indices. Between equations (80) and (82), however, the product of gamma matrices is replaced by a scalar without retaining $\operatorname{tr}\mathbf{1}$. The dimension of the spinor representation is never specified. This is essential: minimal spinors in different dimensions have different numbers of components, and in $2+1$ dimensions the use of two- or four-component fermions is a physical choice. All Dirac results should carry an explicit factor $N_s = \operatorname{tr}\mathbf{1}$ until a representation is chosen.

There is a further independent arithmetic discrepancy. The Bessel integral appearing in equation (84) at $d=1$ is

$$\int_0^\infty \mathrm{d}z\, z^2 \left[K_1(z)^2 - K_0(z)^2\right] = \frac{\pi^2}{16}.$$

Using equation (84) as printed therefore gives $\pi V_1/(8m)$ before correcting the angular measure, not the $\pi V_1/(32m)$ reported in equation (85). Correcting the half-plane angular factor and restoring $N_s = 2$ for a minimal $1+1$-dimensional Dirac field gives $V_1/(4m)$. This agrees with a direct two-band Hamiltonian calculation and does not support the factor-of-four scalar relation asserted after equation (89). The cutoff-dependent coefficients in equations (86) and (88) must also be recomputed with the spinor multiplicity and a regulator matched to the scalar sector. Indeed, the small-R expansion of equation (84) for $d=3$ gives a leading term $4V_3/(3\pi^3 k)$ before the spinor trace, not $4V_3/(3\pi^2 k)$ as printed in equation (88). With a standard four-component Dirac field, this leading radial-cutoff term becomes $16V_3/(3\pi^3 k)$. Thus equation (88) contains an additional factor-of-π error beyond the missing multiplicity.

6. **The massless scalar result does not establish the claimed holographic duality.**

   The Abstract, the paragraph following equation (40), and the Conclusion infer an AdS-volume interpretation from the absence of explicit m dependence in the $d=2$ expression. The cited volume proposal applies under substantially stronger assumptions, in particular to a specified holographic CFT and, for the simple maximal-volume statement, an exactly marginal deformation. The operator ϕ^2 is a relevant deformation of the $2+1$-dimensional free scalar theory, and no large-N holographic theory, bulk field, normalization, or regulator matching is supplied here.

   There is also a parameterization problem at the conformal point. The scalar action depends on m^2, and the conversion in equation (34) uses the reparameterization rule stated in equation (16). The manuscript explicitly assumes there that the map is invertible with nonzero derivative; those conditions fail for $m \mapsto m^2$ at $m=0$. Moreover, the limit $m \to 0$ after the infinite-volume integral need not equal a QFI defined by a differentiable family exactly at $m=0$, especially in the presence of scalar zero-mode and infrared issues. The authors should compute a regulated fidelity with a clearly chosen parameter and order of limits. Unless a genuine holographic setup is developed, the AdS claim should be removed from the Abstract and Conclusion.

7. **The metrological interpretation confuses extensivity and ultraviolet divergences with operational precision.**

   The factor V_d is the expected extensivity of fidelity susceptibility in a translation-invariant many-body system. Sending an accessible region to infinite volume and observing that the total QFI diverges does not by itself define an experiment with perfect mass precision. Likewise, a UV divergence counts arbitrarily short-distance degrees of freedom in an unregulated continuum description; it is not automatically usable information. The paper should report QFI densities at fixed UV and IR regulators and explain the accessible spacetime region, measurement algebra, and resource counting before drawing operational conclusions.

Several QCRLB statements are also inconsistent with the displayed QFI. From equation (40), $I^{(KG)}_{m,3+1} = mV_3/(16\pi)$, so the inverse bound quoted after equation (68) would be $16\pi/(NmV_3)$, not $32\pi/(NmV_3)$. Equation (74) starts with $1/(2m^2)$, whose inverse gives $\mathrm{Var}(\hat{m}) \geq 2m^2/N$, whereas the following paragraph states $1/(4Nm^2)$ and then inverts yet another expression. These errors are dimensionally visible and must be corrected.

Finally, the statement in Section III that an apparent violation of the calculated QCRLB would "falsify any QFT" is too strong. The bound is local and conditional on the state family, regulator, parameterization, resource count, measurement access, and estimator assumptions. A discrepancy would test that entire model of preparation and inference, not uniquely falsify the underlying QFT. The extension from the vacuum construction to "any state" also requires a separate density-matrix or boundary-state derivation.

8. **The novelty claim should be narrowed and supported by a direct comparison with fidelity-susceptibility literature.**

   The Introduction states that no previous work has obtained the QGT of a nonconformal QFT state with respect to a Lagrangian parameter. Yet the object computed here is a single real diagonal component for a pure ground state, namely the ground-state fidelity susceptibility up to a factor of four. That subject has an extensive literature, including the works already cited as Refs. 16, 20, 21, and 45 in the manuscript. Massive Dirac Hamiltonians and their mass-tuned fidelity susceptibility, including the logarithmic $d = 2$ behavior, have also been studied previously in the quantum-criticality literature.

   The authors should distinguish carefully between QGT, QFI, and fidelity susceptibility; survey prior continuum and lattice calculations of massive Gaussian scalar and Dirac vacua; and state exactly which coefficient or renormalized interacting quantity is new. The free results are useful benchmarks but are essentially mode-by-mode Gaussian calculations. The interacting result could supply the main novelty only after the loop and renormalization problems above are resolved.

Minor comments

1. The multiparameter discussion around equations (13)–(15) should note that the SLD matrix bound is not generally jointly attainable when the optimal measurements for different parameters are incompatible. The paper ultimately treats only a single parameter, so the extended textbook review could be shortened considerably.

2. Section I.B appears to reverse the $\tau \to \pm\infty$ assignments of $|\Omega_0\rangle$ and $|\Omega_1\rangle$ relative to the preceding half-space setup. The state-preparation conventions should be stated once and used consistently.

3. Equation (37) defines the gamma function with mismatched integration variables, $s^{z-1}e^{-t}\,\mathrm{d}s$.

4. Figure 1 describes x as "before" and x' as "after" the mass perturbation. More precisely, they are operator insertions on opposite Euclidean half-spaces; the fields are evaluated in the undeformed vacuum in the perturbative correlator.

5. In the $0 + 1$-dimensional discussion, m is the oscillator frequency in the displayed normalization, not the mechanical particle mass. Those concepts should not be identified without introducing an independent mechanical mass. Similarly, the zero Dirac QFI in equation (89) applies only

within a fixed-sign, gapped mass sector; at $m = 0$ the gap closes and changing the sign of m changes the occupied vacuum.

6. The equality step in equation (67) drops an explicit factor of λ. This does not rescue the argument, which already fails because equation (66) is incorrect, but it should be fixed.

7. Appendix B uses $(2\pi)^4$ for one Fourier measure in a calculation presented for general d; it should read $(2\pi)^{d+1}$. The Schwinger parameter integrals there should also show the lower limit 0.

8. The notation in the Introduction alternates between an n-dimensional parameter manifold and a vector ending at θ_m. There is also a duplicated citation to Berry in the discussion of the Berry curvature.

9. “CCRLB” immediately before equation (14) should be “QCRLB”, and the reference to equation (84) in Section II.C is missing a closing parenthesis. Please also correct “he volume”, “lower bounded”, “systems with discrete modes”, and the unmatched parenthesis following equation (74).

10. A revised manuscript would benefit from a compact table listing, for each theory and dimension, the spinor multiplicity if applicable, the regulated QFI density, its divergence class, the renormalization prescription, and which part of the result is universal.