Referee Report

Extended Phase Space Thermodynamics and Joule–Thomson Expansion of Regular AdS Black Holes in a String Cloud

Recommendation: Reject in the present form, with a new submission possible after a complete thermodynamic rederivation. A regular AdS geometry combining an exponential core with a string cloud is a reasonable setting in which to study black-hole chemistry. The present manuscript, however, does not establish a thermodynamically consistent system. Most decisively, equations (21)–(23) imply $(\partial M/\partial S)_{P,\epsilon,r_0} = T/\Psi(r_+)$, whereas equation (27) identifies this derivative with the Hawking temperature T. Thus the first law and Smarr formula used to interpret the volume, Gibbs free energy, coexistence line, heat capacities, and Joule–Thomson expansion fail whenever the regular core is present. The use of three inequivalent specific-volume definitions and the reversal of the heating and cooling regions in Figure 11 are independent problems. These issues affect the central results rather than their presentation alone.

Synopsis and overall assessment

The manuscript begins with Einstein gravity and a negative cosmological constant, supplemented by a Letelier string cloud and a phenomenological effective source. Equations (7)–(10) prescribe a mass-dependent density and integrate it to $\mathcal{M}(r) = (M + \epsilon r/2)(1 - e^{-r^3/r_0^3})$. The resulting metric (11) approaches the string-cloud Schwarzschild–AdS form at large radius and has finite scalar polynomial curvature invariants at the center. Section 2 lists limiting cases, gives sample horizon radii in Table 1, and illustrates the corresponding spherical horizons in Figures 1 and 2.

Section 3 solves the horizon equation for the mass (21) and computes the surface-gravity temperature (22). It adopts the area entropy (23), defines the pressure conjugate (25), and states an extended first law (27) and Smarr formula (29). Section 4 rewrites the temperature as the equation of state (30), introduces an effective specific volume (31), and reports numerical critical points in Tables 2 and 3. The paper attributes the invariance of a critical ratio under changes of r_0 to scale invariance and obtains the usual mean-field critical exponents from the assumed local expansion (34).

Section 5 uses $G = M - TS$, a plotted swallowtail, and a stated Clapeyron relation to infer a small/large black-hole coexistence curve. It also plots the area-law heat capacity (36). Section 6 traces constant-M curves, defines the Joule–Thomson coefficient by equation (37), and displays inversion curves for varying ϵ and r_0. The conclusions emphasize the finite limiting thermodynamic volume, scale-invariant critical ratio, mean-field universality, and an enlarged cooling region.

The paper is organized in a familiar sequence and many of the geometric formulas through equation (22) can be checked directly. The subject is potentially suitable for PRD or JHEP. Its prospective significance is nevertheless modest unless the authors first distinguish their construction from the regular string-cloud solutions and corrected regular-black-hole thermodynamics already cited in the paper. The advertised scale behavior is largely dimensional homogeneity, and the critical exponents follow from a generic cubic Landau expansion. A reliable phase diagram for a genuinely consistent ensemble could still be useful, but the current thermodynamic foundation does not support the manuscript's main claims.

Major comments

1. **The effective theory and its admissible variations are not defined.** Equation (1) contains an unspecified $\mathcal{I}_{\text{eff}}$, while equation (7) simply prescribes an effective density that itself contains the integration constant M. No matter fields, equations of motion, or boundary conditions are given whose

on-shell stress tensor is (7). This matters especially when M, ϵ, and r_0 are varied: a variation of M also varies the matter profile throughout spacetime, so the usual vacuum Einstein first law cannot be imported without an additional matter contribution. The authors should either supply a covariant matter model and derive its covariant phase-space charges, or explicitly formulate the construction as an effective family and derive the corrected mechanical first law for that family. The relevance of Ref. [24], whose title explicitly concerns a corrected first law for regular black holes, should be confronted rather than merely cited in the introduction.

2. **The claimed smooth regular core is overstated.** Equation (12) contains the odd radial term $-\epsilon r^3/r_0^3$, and the text correctly observes that the metric is only C^2 at the origin for $\epsilon > 0$. It then says that the invariants (13)–(15) remain "finite and smooth," which does not follow from their finite limits. For example, applying the scalar-curvature formula to (12) gives

$$R(r) = \frac{24M}{r_0^3} - 32\pi P + \frac{20\epsilon}{r_0^3} r + O(r^2).$$

As a rotationally invariant function of Cartesian coordinates, the term proportional to $r = |\boldsymbol{x}|$ is not differentiable at the center. The authors must state the differentiability standard meant by "regular," examine the stress tensor in a nonsingular frame, and establish that the spacetime admits the required extension and is geodesically complete. Finiteness of three scalar limits alone is insufficient. The distinction between the smooth $\epsilon = 0$ core and the C^2 string-cloud core should be explicit throughout.

3. **Equations (23), (27), and (29) are mutually inconsistent.** This can be seen without any numerical calculation. Differentiate the horizon equation $f(r_+; M, P, \epsilon, r_0) = 0$ at fixed P, ϵ, r_0. Since $f_M = -2\Psi/r_+$ and $f_{r_+} = 4\pi T$, equations (21) and (22) give

$$\left(\frac{\partial M}{\partial r_+}\right)_{P,\epsilon,r_0} = \frac{2\pi r_+ T}{\Psi(r_+)}.$$

With the area entropy $S = \pi r_+^2$ in equation (23), it follows that

$$\left(\frac{\partial M}{\partial S}\right)_{P,\epsilon,r_0} = \frac{T}{\Psi(r_+)},$$

not T. Euler homogeneity correspondingly gives $M = 2TS/\Psi - 2PV + \mathcal{K} r_0$, rather than equation (29), if the derivatives in the manuscript are retained. The authors must decide whether to keep the Einstein-gravity area entropy and introduce the appropriate correction factor or matter work term, or instead to define a different thermodynamic entropy and justify it microscopically. All conjugates must then be recomputed from one integrable first law. The sentence before equation (23) is also self-contradictory: it says the entropy deviates from the area law and immediately imposes that law.

4. **The volume and critical ratio are not defined consistently.** Equation (25) gives a thermodynamic volume $V = 4\pi r_+^3/(3\Psi)$. A volume-based specific length would therefore be proportional to $r_+/\Psi^{1/3}$. Equation (31) instead defines $v_{\text{eff}} = 2r_+[1 - r_+\Psi'/(3\Psi)]$ solely to absorb the coefficient of T in equation (30), while Tables 2 and 3 abandon both choices and use $2r_c$ in ρ_c. Calling the latter a "proxy" does not make the resulting number a thermodynamic compressibility ratio. The authors must identify the variable conjugate to P, use it in the equation of state and Maxwell construction, and recompute every quoted ratio. They should also explain that the r_0-independence in Table 3 follows immediately from $r_c \propto r_0$, $T_c \propto r_0^{-1}$, and $P_c \propto r_0^{-2}$; by itself this dimensional cancellation is not a new universality statement. Finally, the Schwarzschild–AdS limit in Section 2.1 has no finite Van der Waals critical point, contrary to the manuscript's characterization of that limit.

5. **The phase-equilibrium construction must be redone after the first law is fixed.** When equation (27) fails, $G = M - TS$ is not shown to satisfy $dG = -S\,dT + V\,dP + \cdots$. A swallowtail in the plotted algebraic function therefore does not establish global thermodynamic preference. Moreover, the manuscript says that a Maxwell construction is necessary but does not display it, and it plots P against the nonconjugate v_{eff}. Figure 8 provides a smooth coexistence curve without reporting the paired small- and large-hole radii, equal free energies, equal-area residuals, or uncertainties. For representative points, the revised paper should tabulate r_s, r_l, T, P, verify equality of the correctly derived potential, perform the equal-area construction using the pressure-conjugate volume, and numerically check equation (35). The raw numerical procedure and sufficient data for reproduction should be given.

6. **The mean-field exponents are asserted rather than derived for this model.** Equation (34) is the generic cubic form that already encodes the standard exponents, but the manuscript supplies none of its coefficients and does not demonstrate that B and C are nonzero and positive over the stated parameter range. It also uses v_{eff} as the order parameter even though that variable is not conjugate to the pressure. After defining a consistent ensemble, the authors should write A, B, C in terms of derivatives of the actual equation of state, verify the nondegeneracy conditions at each critical point, and derive the coexistence branches from the proper Maxwell condition. The conclusion should then be limited to generic ordinary critical points; the present calculation does not constitute a model-specific proof of a universality class.

7. **The Joule–Thomson section contains both a thermodynamic and a sign error.** Equation (37) is a standard identity only when M is an enthalpy obeying the stated first law. Using the manuscript's own area-law C_P, volume (25), mass (21), and temperature (22), a direct derivative along $M =$ constant instead gives

$$\left(\frac{\partial T}{\partial P}\right)_M = \Psi(r_+)\,\frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right].$$

Thus equation (37) is missing a factor even before the underlying first law is repaired. Although this positive factor leaves the zero set unchanged, Figure 11 and its caption reverse the physical regions. On the left side of each isenthalpic maximum the plotted slope dT/dP is positive, so an expansion $dP < 0$ cools the hole; these points lie *above* the dashed inversion curve. Points below the curve lie on the negative-slope side and heat during expansion. The authors should derive parametric formulas for $P_i(r_+)$ and $T_i(r_+)$, state the physical domain, correct Figures 11 and 12, and recompute the inversion curves from the revised first law.

8. **The physical black-hole branches have not been identified.** The same geometry can have inner, outer, or degenerate horizons, as Table 1 emphasizes. Nevertheless, the later plots parameterize every quantity by a positive root called r_+ without showing that it is the outermost root for the reconstructed M and P. Negative-temperature segments and roots hidden behind a larger zero of f must not enter an equilibrium phase diagram. For every branch used in Figures 3–12, the authors should impose $T \geq 0$, positive pressure where claimed, the desired mass range, and the outer-horizon condition $f(r) > 0$ immediately outside the root. A branch and horizon-domain plot is needed before statements about stable remnants or small/large phases can be assessed.

9. **The novelty relative to the cited literature needs a sharper demonstration.** The paper cites earlier regular black holes in string clouds, extended thermodynamics of regular AdS holes, and Joule–Thomson analyses of regular geometries. It should compare its mass profile, source, corrected first law, and phase structure directly with Refs. [24], [26], [29], [31], [32], and [41], identifying which result is genuinely new. Dimensional scaling with r_0, a standard swallowtail plot, and generic mean-field exponents are not by themselves sufficient novelty for a high-energy theory journal. The paper

would be substantially stronger if it isolated a robust effect of the simultaneous core and string-cloud sources that survives the corrected thermodynamics.

Minor comments

1. In the discussion below equation (25), the correct small-r_+ expansion is $e^{-r_+^3/r_0^3} = 1 - r_+^3/r_0^3 + \cdots$, or equivalently $\Psi = r_+^3/r_0^3 + \cdots$. The displayed limiting value (26) follows from the latter, not from the expansion printed in the text.

2. The captions of Tables 2 and 3 call r_c the "critical radius r_0." These are different quantities. The captions should also state the numerical method and precision rather than using repeated qualitative claims such as "strictly."

3. Figure 2 shows round two-spheres of different radii. Because the ansatz (5) is spherically symmetric, neither ϵ nor r_0 deforms the intrinsic horizon shape; they change its radius and area. The caption and the discussion after the figure should be corrected.

4. The two paragraphs following Figure 3 repeat the description of the same temperature curves. They should be consolidated, and the extremal radius and mass should be quoted numerically for each pressure shown.

5. The range $0 \leq \epsilon < 1$ is stated but not derived. Explain the geometric or energy-condition reason for the upper bound, and discuss the asymptotic normalization associated with the constant deficit $f(r) \sim 1 - \epsilon - 2M/r + 8\pi P r^2/3$.

6. The statement that a string cloud creates "repulsive gravitational effects" is not established by $T^t{}_t = T^r{}_r < 0$. Replace such causal explanations in Sections 4 and 5 by deductions from the displayed response functions, or support them with an explicit force or geodesic analysis.

7. The manuscript should distinguish a finite scalar-polynomial curvature core from a fully smooth regular spacetime, and should not call the limiting thermodynamic volume a geometric buffer that by itself "strictly forbids" collapse.

8. The abstract contains the unexplained phrase "significant geometric inconsistencies" as the purported cause of the critical exponents. More generally, words such as "strictly," "perfectly," "flawlessly," and "rigorously proved" are used where no corresponding check is shown. A thorough technical and language edit is required.

9. Use one notation, v_{eff}, rather than the empty subscript v in equations (31) and (32). Define SBH and LBH on first use, and use "critical point" rather than "crucial point" below Figure 8.

10. The bibliography should give complete publication information for Ref. [43] or identify it as an arXiv preprint. Journal, volume, year, and page formatting should be standardized throughout.

Publication assessment

The geometry may provide a useful phenomenological example, but the paper's principal contribution is its extended thermodynamics, and that analysis is not consistent with equations (21)–(23). Repairing the first law changes the meaning of the pressure conjugate, free energy, coexistence construction, and Joule–Thomson identity, while the horizon-domain and regularity questions require additional work. These are too extensive for a routine revision. I therefore cannot recommend publication in PRD or JHEP in the present form.