Referee Report

on "Analytic Boundaries of Infinite-Spin-Tower Amplitudes from Hidden Zero"

Synopsis and overall assessment

The manuscript studies four-point meromorphic amplitudes constrained by a hidden zero, a higher-point splitting relation, crossing symmetry, polynomial boundedness, and tree-level partial-wave positivity. In the color-ordered setting, the five-point splitting relation is reduced to the multiplicative cocycle equation (5). Its proposed general meromorphic solution is the coboundary form

$$A(s,t) = C\,\frac{f(s)f(t)}{f(s+t)}$$

in equation (6). After restricting the divisor and the large-complex-energy growth, the paper specializes this expression to the finite or infinite product (7). Every massive pole then carries an infinite tower of spins, while an evenly spaced infinite spectrum reduces to the Veneziano amplitude.

The central proposed boundary is obtained in two stages. First, positivity of a residue for $0 < t < \mu_1$ gives the necessary spacing condition $\mu_{k+1} - \mu_k \geq \mu_1$. With $\mu_1 = 1$, the EFT coordinates in equation (12) become moments of the pole locations. Appendix B optimizes the third moment at fixed second moment over spectra satisfying the spacing condition. The maximizing spectrum is $\{1, 2, \ldots, n, r\}$, with $r \geq n + 1$, and gives the piecewise curve

$$X = \frac{H_n^{(3)} + (Y - H_n^{(2)})^{3/2}}{Y}$$

in equation (15). Second, Section IV proves four-dimensional partial-wave positivity analytically for evenly spaced finite products. For the movable-r family, positivity is proved at the first n poles, and also at the final pole for $n = 1, 2$. For general n, the remaining final-pole condition is supported by a finite numerical scan.

The manuscript also derives candidate critical spacetime dimensions for the evenly spaced products, gives the exact level-three scalar coefficient in equation (42), and studies a fully crossing-symmetric triple-product analogue in equation (44). The latter interpolates to the Virasoro–Shapiro amplitude for an infinite evenly spaced spectrum. Appendices D and E discuss its four-dimensional positivity and its spin-zero-subtracted EFT data, respectively.

There is a valuable paper inside the present submission. In particular, the factorization of the evenly spaced residue in equation (17), together with the positive Q_j-function expansion (19), gives a concise analytic unitarity argument in four dimensions. The moment optimization leading to equation (15) is also useful, conditional on the stated spectral class. The explicit $n = 1, 2$ movable-pole arguments and the scalar-subtracted trajectory provide further concrete results. These constructions are well motivated by the numerical splitting region and by the search for closed-form amplitudes near extremal S-matrix boundaries.

The main difficulty is that the strongest claims exceed what is actually proved. The curve (15) is an exact extremum over spectra obeying a *necessary* spacing condition, but general unitarity on that curve is not established. The missing condition is tested only for $3 \leq n \leq 10$, a discrete mesh in r, and spins through $J = 200$. Consequently the paper does not yet prove an analytic unitary boundary, an optimal rule-in region, or that the residual gap must be populated by branch-cut amplitudes. The global classification behind equation (7) also needs a sharper statement of its divisor and complex-growth assumptions.

Several formula and convention problems, including the incorrect recurrence (38) and the malformed product identity (D4), must be corrected before the numerical and positivity arguments can be assessed reliably. I therefore recommend **major revision**.

Assessment of the principal derivations

The strongest analytic step is Section IV.A. With the physical residue defined as the coefficient of $(k-s)^{-1}$, equation (17) factorizes it into the Veneziano polynomial $V_k(t)$ and a rational correction $C_{n,k}(t)$. Under $t = k(\cos\theta - 1)/2$, each correction factor has the positive Legendre expansion (19). Since the linearization coefficients in a product of Legendre polynomials are nonnegative, the argument proves positivity of the evenly spaced family in four dimensions, provided the positivity of V_k is stated in the same convention. This is a clean and potentially useful result.

The boundary optimization is likewise plausible but conditional. The spacing argument following equation (14) correctly identifies an obstruction if a numerator zero lies in $0 < t < \mu_1$. Equations (B4)–(B8) then implement a convexity and summation-by-parts comparison. This proves a moment inequality over the spacing-admissible spectra after one fills a small gap in the proof of $P_K \leq 0$, discussed below. It does not turn the necessary spacing condition into a sufficient unitarity condition.

For the movable pole, equations (20)–(21) establish positivity at the first n poles. Equations (22)–(35) give reasonable direct arguments at $s = r$ for $n = 1, 2$, and the factorwise condition following equation (36) covers sufficiently large r. The uncovered interval for general n is exactly where cancellations among Q_j functions are important. A finite scan there is useful evidence, but it cannot support the theorem-level language presently used for the entire blue curve.

Major comments

1. **The advertised analytic boundary is presently only a candidate unitary boundary.**
   Equation (15) and Appendix B optimize X at fixed Y over spectra that obey $\mu_{k+1} - \mu_k \geq 1$. The manuscript itself correctly derives this spacing condition only as necessary. For the maximizing spectrum $\{1, \ldots, n, r\}$, Section IV.B then leaves the residue at the movable pole unproved for general n in the interval $n + 1 < r \lesssim 1.718n$. The reported check covers only $3 \leq n \leq 10$, r in increments of 0.1, and $J \leq 200$. It neither controls continuous r, arbitrary n, nor the infinite large-spin tail.
   This logical gap affects the title, abstract, Introduction, Figure 3, Section III, and the conclusion. In particular, it is not currently established that every point on the blue curve is unitary, that the blue region is a rule-in region, or that it is the largest meromorphic unitary region. The authors should either prove positivity at $s = r$ for all n, r, J, or consistently call equation (15) the *spacing extremum* and the corresponding blue curve a *candidate* unitary boundary. One possible proof strategy would combine a rigorous large-J asymptotic sign analysis with a finite certified check for the remaining spins. If only numerical evidence is retained, the data and code should cover the continuous r intervals with interval bounds rather than a point mesh.
   The interpretation of the gap with the numerical exclusion curve must also be revised. Until the rule-in curve is established and the exclusion curve is controlled against truncation, branch cuts and finite accumulation points are interesting possibilities, not the only possible explanations.

2. **The passage from the cocycle solution to the "most general" IST product needs a precise classification theorem.**
   Appendix A gives a useful logarithmic-derivative derivation of the local coboundary form, but the global meromorphic statement in equations (A2)–(A7) needs more care at zeros and poles. In

particular, the argument that h has integer residues and hence that the primitive in (A6) exponentiates to a single-valued meromorphic f should be stated for generic slices and completed globally. The divisor assumptions that exclude zeros of f, and hence additional $s+t$-channel poles, should be explicit rather than folded into the phrase "the only meromorphic amplitude we can write down." Equation (A9) displays the essential remaining freedom:

$$A(s,t) \longmapsto e^{q(s)+q(t)-q(s+t)} A(s,t).$$

For example, $q(z) = -\beta z^2/2$ produces $e^{\beta st} A(s,t)$. The authors should specify the exact complex domain and uniformity of the polynomial bound that exclude this deformation. A physical Regge bound on one real ray is not automatically the same as a polynomial bound in all complex directions. The growth hypothesis must be strong enough to justify the claim that pole locations are the only continuous data.

For infinite spectra, convergence and normalization should be stated separately. The natural normalized factor has $C_N = \mu_N$; the statement after equation (8) that C_N must equal N is true only for the special spectrum $\mu_N = N$. The convergence of the ratio product follows from an appropriate condition such as $\sum_N \mu_N^{-2} < \infty$, which is implied by the unit-gap assumption but should not be silently assumed before that condition is derived. These points are central because equations (12), (15), and the completeness claim all depend on equation (7) being exhaustive.

3. **The numerical unitarity claims require correction, certification, and reproducible scope.** The three-term identity printed as equation (38) is wrong. The standard recurrence is

$$(j+1)Q_{j+1}(x) = (2j+1)xQ_j(x) - jQ_{j-1}(x),$$

whereas equation (38) has Q_{j+1} in the middle term. Since this recurrence is presented as the mechanism used to generate all lower-spin values from Q_{200} and Q_{199}, the authors must state whether the implementation used the correct relation and rerun the calculation after fixing the manuscript.

More generally, cancellations among several exponentially small $Q_j(x_0)$ terms are precisely the delicate part of the movable-pole test. Reporting "working precision 100" is not an error analysis. Please provide the code, the definition and normalization of every coefficient, the minimum coefficient or certified lower bound in each tested region, and a stability study under increased precision and spin cutoff. A rigorous interval evaluation would be preferable.

The same concern applies to Table 1. Equation (42) analytically determines the zero of the level-three scalar coefficient, but the statement that this coefficient is always the first one to become negative is based on an unspecified scan over infinitely many partial waves. The ranges in n, k, J, d, the projection algorithm, and an error estimate are needed. Absent an analytic or certified large-spin argument, the tabulated values should be described as candidate critical dimensions supported by the reported scan, and six decimal places are not justified.

4. **Residue conventions and the fully crossing-symmetric positivity proof must be made internally consistent.**

In the color-ordered discussion, equation (17) is the positive coefficient of $(k-s)^{-1}$, which is the negative of the mathematical residue at $s = k$. In Appendix D, equation (D1) instead agrees with the coefficient of $(s-k)^{-1}$ for the fully crossing-symmetric amplitude. This switch is not stated. It also controls the plus sign in the scalar subtraction (E4), because $1/(k-s)$ has mathematical residue -1. The paper should define the physical pole coefficient once in each convention and use notation that does not call two oppositely oriented coefficients $\mathrm{Res}_{s=k}$.

Equation (D4) is also not a valid identity as printed: its factor $1/z_l$ sits outside a product while still carrying the product index, and the numerator factors from (D2) are absent. The factorwise statement needed for the proof is

$$\frac{z_l^2-1}{z_l^2-x^2}=\frac{z_l^2-1}{z_l}\sum_{\substack{J\geq 0\\ J\ \text{even}}}(2J+1)Q_J(z_l)P_J(x).$$

Applied to every l, this does give nonnegative even-spin coefficients, and multiplication then preserves positivity. Appendix D should be rewritten with this correct formula and with all positive normalization factors retained. The special cases $k=0,1$ and the residue orientation should be checked explicitly in the same convention.

5. **Appendix B contains a fixable but important gap in the moment-bound proof.**
After defining P_K in equation (B6), the text says that $\nu_k \geq k$ immediately implies $P_K \leq 0$ for all K. This reason is sufficient only for $K \leq n$, where the comparison spectrum has $\mu_k = k$. At and beyond the movable entry $\mu_{n+1}=r$, the conclusion instead follows from the fixed total moment:

$$P_K=-\sum_{k>K}\nu_k^{-2}\leq 0 \qquad (K\geq n+1),$$

after padding the finite maximizing spectrum by poles at infinity. The proof should be split into these two cases.
The authors should also state the convergence hypotheses needed to pass to $K\to\infty$ in (B7), and give the equality conditions. Strict convexity in (B4), together with equality in the partial-sum comparison, should establish whether the maximizing spectrum is unique up to poles at infinity. These additions would turn Appendix B into a clear conditional optimization theorem and would help prevent it from being mistaken for a unitarity theorem.

6. **The claimed UV-density consequence in the fully crossing case is not established by the example given.**
Section V states that the spectrum $\{1,2,r\}$ violates unitarity for $r>13$, but it does not identify the first negative level and spin or show the calculation. More importantly, failure of this one three-level family does not prove that every finite spectrum with a large gap fails, nor that a pole sent to infinity requires infinitely many intermediate poles. Other finite pole placements can alter all residues.
The negative coefficient behind the $r>13$ statement should be displayed or supplied in reproducible form. A general claim about UV spectral density requires a theorem or a systematic class-wide analysis. Without one, the text should say only that the example demonstrates that the necessary nearest-neighbor spacing condition is not sufficient in the fully crossing-symmetric problem and suggests an additional upper-gap constraint. The connection to the graviton-pole dispersion analysis is interesting, but it does not by itself prove the stronger statement for the triple-product ansatz.

7. **The novelty relative to the cited amplitude literature should be delineated more explicitly.**
The paper notes that the same IST amplitudes were found recently in the higher-point/parity analysis cited as Ref. [33], while the fully crossing-symmetric construction belongs to the triple-product class of Ref. [32]. It also relies on the numerical splitting region of Ref. [21] and should be read alongside the analytic Veneziano bootstrap of Ref. [22]. At present the comparison is too brief to identify which results are genuinely new.
The revision should separately compare: the product classification (7); the EFT moment coordinates (12); the spacing extremum (15); the analytic four-dimensional positivity proof for evenly spaced finite products; the movable-pole results; the candidate critical dimensions; and the spin-zero-subtracted fully crossing trajectory. A compact comparison paragraph or table would make the significance much clearer. My reading is that the moment optimization and several positivity results are the most distinctive contributions, but their precise overlap with Refs. [21,22,32,33] should be established by the authors.

Minor comments

1. The statement following equation (14) should explicitly restrict the positivity argument to $0 < t < \mu_1$, where the crossed-channel Legendre expansion converges. Above that interval the sign test is not being used.

2. In the footnote after equation (17), explain why the extra factor $t + k = \frac{k}{2}(1 + \cos\theta)$ preserves positivity: it has a nonnegative P_0, P_1 expansion, and the Legendre linearization coefficients are nonnegative.

3. The phrase after equation (36) that each factor is "positive" should say that each factor has nonnegative Legendre coefficients under the stated bound. Pointwise positivity and partial-wave positivity are different assertions.

4. The nonlinear example preceding equation (51) has an inconsistent spectrum. The displayed $\lambda(x) = (\sqrt{1 + 8x} - 1)/2$ has poles at $\mu_N = N(N + 1)/2$, not at $N(N + 1/2)$.

5. Figure 3 and its caption should distinguish the rigorously optimized spacing curve from those segments whose unitarity is analytic, numerical, or presently unknown. A line-style legend would be useful.

6. The caption of Table 1 defines d_c through positivity of all partial waves, while the calculation displayed in the text solves only the level-three scalar condition. The caption should reflect the actual status of that implication.

7. The normalization of C_N, the notation n versus N, and the inclusion or exclusion of the massless pole in sets such as $\{0, 1, \ldots, n, r\}$ should be standardized.

8. The conclusion contains a duplicated fragment after the sentence ending "singularities other than poles." It should be deleted.

9. A careful copyedit is needed. Examples include "Take a cyclically rotation," "analogueue," "$N$and," "Virasora–Shapiro," and several subject–verb agreement errors. Quotation marks and dashes should also be encoded consistently.

10. The numerical figures should state the partial-wave normalization, precision, spin range, and whether plotted zeros in the Virasoro–Shapiro limit are exact or below a numerical threshold.

Recommendation

I recommend **major revision**. The finite-product constructions, the analytic positivity proof for evenly spaced poles, and the conditional moment extremum are interesting and potentially suitable for JHEP or Physical Review D. Before publication, the manuscript must align its boundary claims with the actual unitarity proof, make the meromorphic classification assumptions precise, correct and certify the numerical analysis, repair the residue conventions and displayed identities, and soften or prove the proposed branch-cut and UV-density conclusions.