Referee Report

on "geoSCET: Soft Theorems from Power Counting"

Synopsis and overall assessment

The manuscript develops a soft-collinear effective theory for scalar fields whose two-derivative interactions are encoded by a field-space metric. After a useful review of position-space SCET in Section 2, Section 3 splits the scalar into a soft background and collinear fluctuations using the exponential map. The pullback of the Levi–Civita connection then plays the role of an emergent soft connection. Parallel transport, fixed-line gauge, the multipole expansion, and the structure of the N-jet operator are organized to mirror their gauge-theory counterparts. Equations (3.43)–(3.45) introduce a shift of the reference vacuum and use it to relate hard coefficients with different soft multiplicities.

Section 4 applies the construction to amplitudes. Equations (4.8)–(4.18) reproduce the single-soft geometric theorem for a derivative theory, while equations (4.30)–(4.33) address the simultaneous double-soft limit. The potential sector gives the external-line pole in equation (4.22) and a next-to-soft momentum derivative in equation (4.27). The loop discussion separates hard, collinear, and soft regions, reproduces the one-loop bubble, triangle, and box structures, and proposes an all-orders argument for the tree-level exactness of the derivative-theory soft theorems. Appendices B–D add useful discussions of discontinuity functions, RPI relations, and the infinite soft theorem.

The central idea is novel and potentially significant. In particular, the identification of field-space parallel transport as the scalar analogue of the soft Wilson-line structure gives an illuminating EFT interpretation of the geometric soft theorem of Ref. [10]. The separation of hard matching from low-energy regions also complements the explicit one-loop analysis of Ref. [43]. If established with the claimed scope, the multiple-soft and all-orders results would be of clear interest to readers of JHEP or Physical Review D. The manuscript is generally well organized, and Appendix B provides a helpful concrete bridge between discontinuity functions and regions.

I do not, however, think the principal claims are yet demonstrated at journal standard. There is a sign inconsistency in the soft-decoupling Wilson line, the RPI formula used in the main text does not agree with its purported all-order derivation, and the double-soft hard vertex appears to have an uncancelled symmetry factor. More broadly, the "all orders" statement does not address the higher-derivative counterterms required by a four-dimensional nonlinear sigma model, while the conclusions for a cubic potential rely on a correlated scaling of the coupling and soft momentum rather than the ordinary soft limit of a fixed theory. The massive-loop conclusion is asserted before the relevant integrals are defined or calculated. I therefore recommend **major revision**. The framework is promising, but the points below should be resolved before the paper is suitable for publication.

Assessment of the logical structure and central derivations

The background-field expansion through quadratic order is standard and, modulo convention checks, equations (3.12)–(3.17) have the expected covariant form: the fluctuation is a tangent vector, its kinetic operator contains the pullback connection, and the curvature controls the background-dependent quadratic term. The exact metric-transport identity in equation (A.39) is also a useful way to organize the multipole expansion. These ingredients make the proposed geoSCET construction credible.

The tree-level single-soft calculation is likewise structurally sound. The external-line insertion in equations (4.8)–(4.11) produces the connection term, and the background dependence of the hard coefficient supplies the partial derivative in equation (4.13). Their sum has the tensorial form displayed in equation (4.16). The curvature contribution to the double-soft theorem in equation (4.31) also has the expected antisymmetric momentum weight. These are important checks of the construction.

The logical difficulty is that several steps used to promote these checks into general theorems are either coordinate dependent or algebraically inconsistent. The ordinary split $\varphi = v + \varphi_s$ in equation (3.34) is precisely the kind of addition of field-space points that Appendix A correctly rejects for the hard/soft background split. Equations (3.43)–(3.45) therefore do not by themselves define a field-space-covariant symmetry. In addition, equations (3.46)–(3.49) do not solve the decoupling equation with the sign convention for D_μ used earlier. Finally, equation (4.6) and equation (C.6a) are inequivalent for generic N. Since these elements are subsequently invoked in the single-soft, double-soft, and all-multiplicity arguments, they must be repaired rather than treated as notational details.

Appendix B verifies two one-loop region expansions, but this does not yet supply the all-loop infrared theorem claimed after equation (4.34), nor the massive factorization conclusion following equation (4.49). The distinction matters: the former requires control of arbitrary nested soft/collinear subgraphs and counterterm insertions, while the latter requires a regulator-complete analysis of lightlike eikonal integrals. The manuscript currently moves from selected one-loop examples to conclusions that are substantially stronger than those examples establish.

Major comments

1. **The hard-coefficient "shift symmetry" must be formulated covariantly.** Equations (3.43)–(3.45) use $v + \varphi_s$ and a constant coordinate shift a. Neither operation is intrinsic on a curved field space. Moreover, $C_{I_1 \ldots I_N}(\varphi)$ carries flavor indices in tangent spaces at the point φ, so coefficients evaluated at different base points cannot be compared without parallel transport. An ordinary Taylor series in partial derivatives is justified in a chosen normal-coordinate chart, but it is not an invariant statement and cannot by itself establish the advertised field-diffeomorphism origin of the theorem. The authors should define the soft displacement as a tangent vector $\eta \in T_v\mathcal{N}$, for example through $\varphi = \exp_v \eta$, parallel-transport all hard-coefficient indices back to $T_v\mathcal{N}$, and derive the covariant Taylor series explicitly through at least quadratic order. This calculation should reproduce equations (4.13), (4.16), and (4.33), including every connection and curvature term. It should also state what precisely is meant by the claimed relation among arbitrary soft multiplicities. Until this is done, the ordinary-coordinate identity in equation (3.45) cannot support the all-multiplicity claim.

2. **The soft-decoupling Wilson line has the wrong sign for the stated covariant derivative.** With $D_\mu = \partial_\mu + A_\mu$, the equality in equation (3.47) requires

$$n\cdot(\partial + A)S_n = 0.$$

The path-ordered exponential in equation (3.46), however, has a plus sign, so with the usual endpoint convention it obeys $n\cdot\partial S_n = +(n\cdot A)S_n$. Substitution into the left-hand side of equation (3.47) leaves two connection terms rather than zero. The solution requires a minus sign in the exponential, or an inverse Wilson line and a correspondingly changed field definition. The signs in equations (3.48) and (3.49) must change as well. The authors should write the endpoint differential equation explicitly and propagate the corrected convention through the decoupling discussion and the covariance-restoration argument following equation (4.33).

3. **The RPI coefficient in equation (4.6) is inconsistent with Appendix C.** Let $p_{i-} = \bar{n}_i\cdot p_i$ and

$S_{ik} = 2p_{i-} \cdot p_{k-}$. Equation (C.6a) gives, schematically,

$$C_i^{A1\,\mu} = -\frac{1}{p_{i-}} \sum_{k \neq i} \frac{2n_k^{\mu\perp_i}}{n_i \cdot n_k} \frac{\partial C^{A0}}{\partial \ln S_{ik}}.$$

By contrast, equation (4.6) places the same total derivative $\partial C^{A0}/\partial p_{i-}$ behind every vector in the sum. For $N > 2$, $p_{i-}\partial_{p_{i-}} C = \sum_l \partial_{\ln S_{il}} C$, so equation (4.6) contains a product of two sums, whereas equation (C.6a) pairs each direction n_k with its own invariant derivative. They coincide only when a single independent invariant contributes, as in a two-direction special case. The claim below equation (C.6) that the formulas agree is therefore not correct as written.

This is important because the $\mathcal{L}^{(1)}$ insertion and the derivation of equation (4.27) rely on the RPI relation. The authors should correct equation (4.6), include the transverse projector, and redo the next-to-soft derivation for a genuinely generic N-jet amplitude. Momentum-conservation constraints and the choice of independent invariants should be stated explicitly.

4. **The double-soft hard-emission normalization is internally inconsistent.** Equation (3.45) contains $\frac{1}{2}\varphi_s^A \varphi_s^B \partial_A \partial_B C$. With standard external-state normalization, the two contractions of the soft fields cancel this Taylor-series factor, giving a hard-vertex amplitude $\partial_A \partial_B C$, not the one-half of equation (4.32). This is also what equation (4.33) requires: in normal coordinates, $\nabla_{(A}\nabla_{B)} C = \partial_A \partial_B C$ under the usual normalized symmetrization convention, not $\frac{1}{2}\partial_A \partial_B C$. The authors should derive the two-soft Feynman rule with explicit flavor sums and state their symmetrization convention. As printed, equations (4.32) and (4.33) cannot both follow from equation (3.45).

5. **Renormalization and higher-derivative operators are missing from the all-orders claim.** The two-derivative nonlinear sigma model in four dimensions is not closed under renormalization: loops generate covariant higher-derivative counterterms. Thus the statement after equation (3.40) that the tree-level geoSCET Lagrangian is exact can at most mean that the displayed structures require no additional hard matching. It cannot mean that no new operator structures are required. This is especially relevant when equations (4.34)–(4.35) are used to claim an all-orders theorem for a "general scalar EFT."

   The paper should specify both expansions being used: the SCET expansion in λ and the derivative expansion relative to the scalar-EFT cutoff. It should then include the most general covariant higher-derivative operators and their counterterms, or prove that their insertions are parametrically incapable of contributing at the leading single- and multiple-soft orders. A complete all-orders argument should address arbitrary soft and collinear subgraphs, overlap/zero-bin subtractions, operator mixing, and possible exceptional kinematics. The observation that a one-loop graph needs another $\mathcal{O}(\lambda^2)$ vertex is a useful first step, but it is not by itself a proof for every loop order and every number of soft legs. If such a proof is outside the intended scope, the claims after equation (4.35) and in the abstract should be weakened accordingly.

6. **The potential analysis uses a correlated double scaling, not the ordinary soft limit at fixed coupling.** Equation (4.36) identifies the loop parameter as $g_3^2/(p \cdot k_s)$ and declares it of order λ^2 by assigning $g_3 \sim \lambda^2 Q$. For a fixed theory, however, g_3 does not tend to zero when $k_s \to 0$; the displayed ratio instead diverges, exactly as the following text observes for higher loops. Statements that loop corrections are subleading therefore apply only to the correlated regime $g_3^2 \ll p \cdot k_s$, or to the massive high-energy expansion with a specified relation among g_3, m, and Q. They are not statements about the strict massless soft limit at fixed g_3.

   There is also a vacuum issue in equations (4.19)–(4.27). A massless stationary point with a nonzero real cubic tensor is generically a saddle: along some field-space direction the cubic term is negative, and a quartic term cannot stabilize the potential arbitrarily close to the origin because it is subleading there. Hence the proposed fine-tuned massless theory need not admit the stable asymptotic vacuum required for its S-matrix. The authors should state clearly that this is a formal double-scaling example, or replace it with a well-defined massive setup. They should also display the complete potential soft relation, adding the covariant k_s^0 term of equation (4.16) to the pole and momentum-derivative terms in equations (4.22) and (4.27), and distinguish flavor from mass eigenstates.

7. **The massive-loop conclusion is not established by equation (4.49).** The schematic integral in that equation contains one lightlike eikonal denominator. A mass supplies a virtuality scale, but it does not by itself regulate the homogeneous boost/rapidity integration associated with $n \cdot \ell$. In dimensional regularization alone, such an integral is not automatically a nonzero object with an ordinary ultraviolet pole; depending on the prescription it is scaleless or requires an additional rapidity regulator, residual energy, or second direction. The missing $i0$ prescription is also essential.
The manuscript first says that the necessary massive topologies are left for future work and then concludes that their infrared finiteness and cancellation are sufficient to prove that equation (4.22) receives no loop correction. That conclusion is premature. The authors should either calculate the complete one-loop massive soft and collinear regions with a consistent regulator and zero-bin prescription, demonstrating the claimed cancellation, or restrict the conclusion to a power-counting expectation. The statement in Appendix B that dimensional regularization regulates every $\mathrm{SCET_I}$ region should also be reconciled with the lightlike eikonal integral in equation (4.49).

Minor comments and presentation

1. Equation (A.16) writes covariant derivatives of a Christoffel symbol as though the connection were a tensor. The generalized connection coefficients in the exponential-map series require a specific recursive definition; already at cubic order the combination of $\partial\Gamma$ and $\Gamma\Gamma$ terms must be stated. Please replace equation (A.16) with a correct definition or limit the displayed expansion to the order actually used.

2. The second line of equation (A.18) contains $\partial_\mu \varphi$ without its field-space index. Several other formulas alternate between A_s and A for the pullback connection. A short notation table would help distinguish the QCD connection, the geometric pullback connection, and their fixed-line-gauge versions.

3. In equation (4.22), the denominator should retain the leg label, $p_{a-} k_{sa+} = 2p_a \cdot k_s$, since each sector has its own light-cone pair. Writing the invariant form would also make the relation to equation (4.27) transparent.

4. In equations (4.25)–(4.26), k_{s+} is used both as though it were a component and as though it were the multipole-projected vector entering the delta function. Define the latter explicitly. The sign convention for $k_{s\perp}^2$ should also be stated before using the on-shell cancellation in equation (4.26).

5. The explicit calculation leading to equation (4.25) should show which momentum derivatives act on the propagator and which act on the hard coefficient. As printed, the coefficient appears outside the differentiated delta function, so the origin of the surviving derivative of C is difficult to follow.

6. Footnote 17 is adequate for a massless degenerate flavor basis, but the potential and massive sections require asymptotic mass eigenstates. Please state the assumptions on the Hessian of V, flavor degeneracies, and the tetrads used to convert the tensors ${V_{AI}}^J$ into physical couplings.

7. The claim of infrared finiteness in Section 3.2 should specify nonexceptional hard kinematics and distinguish finiteness of fixed-angle amplitudes from the behavior of inclusive observables and from limits in which additional hard invariants also vanish.

8. Appendix B uses spacelike invariants in the examples, whereas Section 4 uses physical scattering kine-

matics. Please state the analytic-continuation and branch conventions, including the $i0$ prescriptions, for equations (B.3)–(B.27).

9. The literature discussion should identify the genuinely new statements more sharply. In particular, separate the known background-field expansion, the position-space SCET machinery of Refs. [55–58], the tree-level geometric theorems of Ref. [10], and the one-loop structures of Ref. [43] from the new geoSCET construction and the proposed all-orders conclusions.

Recommendation

The manuscript presents an attractive framework and contains enough original material to merit a serious revision. At present, however, the sign, normalization, and RPI inconsistencies affect derivations used centrally in the paper, and the strongest all-orders and massive-factorization statements extend beyond what has been proved. I recommend **major revision**. A revised version that gives a genuinely covariant hard-coefficient expansion, corrects equations (3.46), (4.6), and (4.32), treats renormalization and the cubic double-scaling honestly, and either computes or qualifies the massive loop sector would be a plausible candidate for publication in JHEP or Physical Review D.