Referee Report

On quantum corrections to semiclassical strings on $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_L$ orbifold backgrounds

Synopsis

The manuscript studies three circular semiclassical type IIB string solutions on $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_L$ and compares their large-charge energy expansions with the dual planar $\mathcal{N} = 2$ circular-quiver spin chains. The orbifold acts with opposite phases on two complex coordinates of S^5, so a string in the nth twisted sector can carry the fractional covering-space winding $\mu = \widetilde{m} + n/L$ introduced in equation (2.6). The paper concentrates on the branch $\widetilde{m} = 0$ and considers: (i) an equal-spin $J_1 = J_2$ solution in the SU(2) sector, equations (2.5)–(2.9); (ii) its three-spin extension with $J_3 \neq 0$, equations (2.10)–(2.17); and (iii) an SJ solution with spin in AdS_3 and angular momentum along an orbifolded circle, equations (2.19)–(2.26).

For each family the manuscript constructs the quadratic bosonic and fermionic fluctuation operators and evaluates the one-loop energy. For the equal-spin solution the result is the convergent coefficient in equations (3.8)–(3.9), with stability $|\mu| < 1/2$. For the three-spin solution the fluctuation frequencies are roots of the quartic (B.7) and the eighth-order fermion polynomial (B.20); the coefficient is then obtained numerically from equation (3.14), and a filling-dependent stability region is derived in equations (3.11)–(3.13). For the SJ solution the result is the sum or integral in equation (3.18). Section 4 derives the corresponding Landau–Lifshitz models and obtains equations (4.11), (4.19), and (4.27). Section 5 adapts twisted Bethe equations to reproduce the full leading finite-size terms in the SU(2) and SL(2) cases, equations (5.7) and (5.9), while in the SU(3) case it compares only the regular contribution (5.11), not the full nearby-root correction. Section 6 finally comments on a long-quiver scaling of L.

Assessment and recommendation

This is a technically ambitious and potentially useful extension of the classic semiclassical-string program to a supersymmetric orbifold with nontrivial twisted sectors. The presentation brings together the world-sheet fluctuation calculation, the Landau–Lifshitz description, and the twisted spin chain in a coherent way. The fractional-winding stability windows, especially for the three-spin solution, are interesting. The subject and level of detail are appropriate for JHEP or Physical Review D.

The novelty should nevertheless be calibrated carefully. The classical orbifold solutions and the SU(2) Landau–Lifshitz/winding-state analysis build directly on Ideguchi and on Astolfi, Grignani, Harmark, and Orselli. The fluctuation technology and the unorbifolded SU(2), three-spin, and SL(2) results are closely related to the work of Frolov, Tseytlin, Park, and collaborators, while the finite-size matching follows Beisert, Tseytlin, Zarembo, Freyhult, and Kristjansen. Much of the SU(2) and SL(2) answer is an analytically continued replacement of an integer winding by μ. The genuinely new content is therefore the global orbifold implementation, the fractional-winding stability analysis, and the numerical three-spin result. Precisely those points need stronger validation.

I recommend **major revision**. I do not think the one-loop claims are ready for publication until the global fermionic boundary conditions are established, the incorrect large-mode formula is repaired, and the three-spin numerics are made reproducible. The full versus partial scope of the gauge/string comparison must also be stated accurately. If these issues are resolved, the paper could become a valuable specialist contribution.

Major comments

1. **The twisted-sector boundary conditions for the Green–Schwarz fermions must be derived globally.**

   The defining monodromy of the nth sector follows from the orbifold action (1.5) and the winding condition (2.6). Bosonic fluctuations can be written in quotient-adapted coordinates so that the background monodromy is factored out. The Green–Schwarz fields, however, are target-space spinors. Their boundary condition depends on the lift of the $\mathbb{Z}_L$ generator to Spin(6) and on the transition function of the chosen orthonormal frame around the closed string.

   The manuscript observes after equation (A.26) that the pulled-back connection is σ-independent and therefore inserts the integer expansion $r \in \mathbb{Z}$ in equation (A.34). Constancy of a local connection does not by itself determine the global spin-bundle transition. The same assumption enters the frequencies (3.5), the three-spin determinant (B.20), and the SJ frequencies (C.20). The author should:

   - write the orbifold action on the two ten-dimensional Majorana–Weyl spinors;
   - give the spin lift and the boundary condition under $\sigma \mapsto \sigma + 2\pi$ for every twisted sector and winding branch;
   - show explicitly whether the chosen moving frame converts that condition to periodic component fields, and how its holonomy is represented in (A.26), (B.16), and the operator leading to (C.20); and
   - explain how alternative spin structures, when available, are excluded or implemented.

   If any components are fractionally or antiperiodically moded, the replacement $r \mapsto r + \nu$ changes the vacuum sum, the ultraviolet pairing, the stability bounds, and the claimed equality with the Landau–Lifshitz and Bethe results. This is a foundational point, not a convention that can be left implicit.

2. **Equation (B.25) does not follow from the characteristic polynomial (B.7).**

   Let $q = \sin^2 \gamma_0$ and substitute $w = |r| + a + b/|r| + \cdots$ into equation (B.7). The leading r^4 coefficient gives

   $$a^4 - 2Ba^2 + \kappa^2 \mathrm{k}^2 = 0, \qquad B = \mathrm{k}^2 + \mu^2(2-q).$$

   Consequently,

   $$a^2 = B \pm \sqrt{4\mathrm{k}^2\mu^2(1-q) + \mu^4(2-q)^2},$$

   and the four constant shifts are the two signs of the square roots of these two quantities. In contrast, equation (B.25) prints terms proportional to $\mathrm{k}^2 + \mu^2(2-q)$ directly as constant frequency shifts, omits the outer square root, and has an incompatible inner μ^4 term. The displayed expression cannot satisfy (B.7) for generic parameters.

   The author should correct (B.25), rederive the sum (B.26), and verify the large-r fermion expression (B.30) in the same conventions. Since ultraviolet finiteness of the paired sum in (3.14) is justified using (B.26) and (B.30), the cancellation should be demonstrated from corrected branch expansions, including the first nonvanishing $|r|^{-3}$ coefficient. The numerical implementation should then be rerun from the correct polynomial, with explicit confirmation that this error was confined to the printed asymptotic formula.

3. **The three-spin one-loop calculation needs a reproducible spectral and numerical prescription.**

   Equation (3.14) is central to the paper, but the information supplied is insufficient to reproduce table 1 or figures 2–3. In particular, equation (B.20) is even in the frequency, while the text selects "signed characteristic roots" by continuity from the large-κ branches (B.31). Near degeneracies and instability thresholds this continuation is not unique without an ordering and path prescription. The author should specify how bosonic and fermionic roots are paired at each r, how roots are tracked as κ, q, or μ varies, how complex-conjugate roots contribute, and how the $q \to 0$ and $q \to 1$ coordinate-degenerate endpoints are taken.

The cutoff $\Lambda = 40000$ and a fit using only $\kappa = 50, 100, 200$ are quoted without tail bounds, residuals, uncertainties, or raw values. Please provide code or pseudocode sufficient to reproduce the calculation, convergence tests in both Λ and κ, and uncertainty estimates for every table entry. Useful analytic checks are the BMN limit $\mu \to 0$, the $q \to 1$ reduction to the equal-spin solution, the $q \to 0$ limit after changing to nonsingular coordinates, and a direct comparison of the leading light branches with equation (4.19). Values close to zero and points close to the stability boundary require particular care.

4. **The SU(3) gauge/string agreement is only partial and is currently overstated.**

   The abstract and parts of Sections 1 and 6 say that the leading quantum correction for each solution is matched to the finite-size Bethe result. In the SU(3) sector, however, Section 5 imports only the regular term (5.11). The anomalous contribution from roots separated by order $1/J_{\text{tot}}$ is explicitly not computed. Thus equations (5.11)–(5.13) test only a zero-mode/regular component; they do not match the full oscillator sum (3.14) or the full Landau–Lifshitz expression (4.19).

   Either the author should calculate the anomalous finite-size term from the twisted nested equations (D.4), or every broad statement of a three-way match should be restricted to the precise component actually checked. It would also help to separate the genuine $r = 0$ string contribution from the finite constants generated when the Landau–Lifshitz sum is regularized: the phrase "zero-mode part of (4.19)" is potentially misleading because the displayed light frequencies vanish at $r = 0$. Finally, the discussion should acknowledge the order-of-limits issue emphasized in the unorbifolded three-spin literature rather than implying that reduced supersymmetry is the only new conceptual obstacle.

5. **The finite part of each Landau–Lifshitz vacuum energy requires a common regularization argument.**

   Equations (4.11), (4.19), and (4.27) are obtained by subtracting large-r terms and assigning values through zeta-function regularization. The finite constants are essential for the claimed equality with the string and Bethe coefficients. In an effective Landau–Lifshitz theory those constants are scheme-dependent unless a matching prescription to the ultraviolet world-sheet theory is supplied. This is especially important on an orbifold, where the global moding discussed in comment 1 can modify the regulated sums.

   Please formulate a single regulator that can be applied before the fast-string limit, show how it reduces to the prescriptions in all three sectors, and identify any local counterterms or normal-ordering constants. The equal-spin case can be anchored to the known contour treatment, but the three-spin and SJ cases still need an explicit argument. Analytic continuation into an unstable region should also be distinguished from a vacuum energy of a stable quantum state.

6. **Stability should be described in terms of the full winding branch, not only the representative $\widetilde{m} = 0$.**

   Equation (2.6) shows that a fixed twisted sector admits all $\mu = \widetilde{m} + n/L$. The footnote after equation (3.5) correctly notes that, for $n > L/2$, choosing $\widetilde{m} = -1$ gives $|\mu| < 1/2$ and reverses the apparent instability of the chosen representative. Nevertheless, several later statements and figure captions describe stability as a property of the twisted-sector label n.

   The author should give a systematic map between $\widetilde{m}$ on the string side and the logarithmic branch integers in the Bethe equations, including equation (5.4) and the two branches in (D.4). The physical spectrum should be organized by $(n, \widetilde{m})$, or by a branch-independent quantity such as the minimal $|\mu|$, and the relation between the n and $L-n$ sectors should be explained. The novelty claim should then distinguish new stable saddles from a relabeling of winding branches.

7. **The long-quiver discussion is outside the controlled semiclassical regime in a stronger sense than stated.**

   In Section 6, $R/\ell_s = \lambda^{1/4}$, whereas the shortest quotient cycle has size of order R/L. Thus in the proposed regime $L \gg \sqrt{\lambda}$ one has $R/(L\ell_s) = \lambda^{1/4}/L \ll 1$. Winding modes and an equivalent dual description cannot be ignored. The observation around equation (6.1) that powers of $1/L$ reorder the fixed-background loop expansion is useful, but it is not by itself a controlled semiclassical prediction. The text should state the additional sub-string-scale obstruction, identify the degrees of freedom expected to become light, and sharply separate the formal scaling exercise from conclusions supported by the calculations at fixed L.

Minor comments

1. Equations (3.10) and (B.27) are assigned the same cross-reference key in the manuscript source. As a result, citations in Sections 4 and 5 can resolve to the wrong printed equation. The keys and all affected references should be corrected.

2. The conditions under which the reference states in equation (1.9) vanish should be explained directly using trace cyclicity and the twist phase. This would make the distinction between a formal Bethe vacuum and a nonzero gauge-invariant operator much clearer.

3. Table 1 includes $q = 1$, although the derivation of (B.7) explicitly removes a factor proportional to $q(1-q)$ and is stated for the nondegenerate branch. Quote the table entry as a controlled limit from nonsingular variables, or omit it.

4. The claim that imaginary parts represent an "energy" should be qualified. In the unstable region they diagnose a negative fluctuation mode and decay of a classical saddle; they are not complex eigenvalues of a stable Hermitian Hamiltonian.

5. Use one notation for the three-spin angular velocity and distinguish it visibly from Bethe mode integers. The simultaneous use of k, k, m, $\widetilde{m}$, p, q, and r makes several derivations unnecessarily difficult to follow.

6. In equation (C.16), the branch of the angle l should be stated when $\cos l < 0$. This matters for the spin rotation and is also relevant to the global boundary-condition issue above.

7. The manuscript would benefit from a compact table listing, for each of the three solutions, its charges, physical closure condition, winding data, stability domain, one-loop coefficient, and the exact extent of the Bethe and Landau–Lifshitz match.

8. Several wording and typesetting points should be corrected throughout: "consists on taking" should read "consists of taking"; "at the level global identifications" is missing "of"; "associated the transverse" is missing "with"; and the spelling and encoding of "Bethe ansatz/ansätze" and accented names should be made consistent.