Referee Report

on "Holography in the linearized quantum gravity regime and modular crossed product"

Synopsis and overall assessment

The manuscript studies a ball-shaped boundary region in the vacuum of an AdS_{d+1}/CFT_d dual pair and the associated AdS–Rindler wedge. Section 2 introduces a family of asymptotically AdS metrics, imposes a Gaussian-null-type gauge, and argues that the vacuum-subtracted HRT area begins at second order in the perturbation. Section 3 describes radiative graviton data on the future AdS–Rindler horizon and relates a boost flux to asymptotic and corner charges. Section 4 identifies the vacuum modular generator with the geometric boost and adjoins an auxiliary charge degree of freedom, producing a crossed-product algebra with a semifinite trace.

Section 5 assumes an isometric holographic map that preserves the vacuum and implements exact wedge reconstruction on a code subspace. The modular and relative modular operators are then transported through this map, yielding an equality of relative entropies. Section 6 specializes to coherent graviton states. It combines a known crossed-product entropy formula, the coherent-state relative entropy in equation (6.8), and a Noether-charge identity at the bifurcation surface to obtain equation (6.15). The manuscript interprets this as a Type-II formulation and proof of the vacuum-subtracted HRT relation. Appendix A discusses the KMS property, Appendix B asserts purification independence of the crossed-product entropy, and Appendix C reviews amenability and a mean ergodic theorem.

The goal is worthwhile. A clean relation among AdS wedge reconstruction, crossed products, and geometric entropy would be useful, and the paper brings together several recent operator-algebraic developments in an accessible way. The use of coherent null data also gives a concrete testing ground. However, the main conclusion is not established. The argument removes the finite bulk entropy from the quantum extremal surface formula, does not justify the absence of a first-order area variation, and uses mutually inconsistent expressions for the boost flux. At the algebraic level, the compressed code-space algebra is assumed to have properties that do not follow from wedge reconstruction, while Appendix B extends equality of states from a Type-III algebra to its crossed product without justification. The final equality also drops the state-independent, wave-function-dependent terms that the preceding formula explicitly contains.

Several of these issues affect the central theorem rather than its presentation. They would require a new formulation of the physical algebra and state, a careful perturbative area calculation, and a revised statement of what is actually proved. I therefore recommend **rejection in the present form**. A substantially reconstructed manuscript could merit reconsideration because the underlying question is interesting, but the current derivation is not yet suitable for JHEP or Physical Review D.

Logical structure, assumptions, and relation to prior work

There are two conceptually distinct arguments. The first is the relative entropy statement in equations (5.11)–(5.14). If one already has a genuine unitary equivalence between the bulk wedge von Neumann algebra and a boundary code-space von Neumann algebra, preserving the reference vector, then Tomita operators, modular operators, and Araki relative entropy are invariant under that equivalence. Subject to domain qualifications, this part is a standard consequence of modular theory. It is not, by itself, a dynamical derivation of JLMS: essentially all of the content has been placed in the exact algebraic identification assumed in equation (5.3). Moreover, the entropy obtained is that of the compressed code algebra, not automatically the entropy of the full boundary subregion algebra appearing in the usual JLMS statement.

The second argument is the proposed HRT result. Its chain of implications is

$$\text{coherent relative entropy} \longrightarrow \text{weighted horizon flux} \longrightarrow \text{boundary-minus-corner charge} \longrightarrow \text{second-order area.}$$

The first arrow is closely related to known coherent-state relative-entropy formulas, and the last two are modeled on the Iyer–Wald identity and on recent crossed-product treatments of horizon entropy, including the author's earlier work. The new claim is that this chain gives the boundary HRT entropy in AdS. That is precisely where additional ingredients are required: the correct semiclassical generalized entropy, a physical constraint relating the auxiliary charge to a second-order gravitational completion, and control of the first variation of the extremal area.

The manuscript cites the foundational RT/HRT, FLM/QES, JLMS, and crossed-product literature, as well as recent results on generalized black-hole entropy. The positioning should nevertheless be sharpened. The crossed-product entropy formula and the coherent horizon relative entropy are imported results; the relative-entropy equality follows formally from the assumed exact algebra isomorphism. Thus the novelty rests on the AdS charge/area identification and its boundary interpretation. Until the gaps in those steps are resolved, the claims of a rigorous proof and of validity for every $d \geq 2$ are too strong.

Major comments

1. **Equations (1.2)–(1.4) do not give the semiclassical QES formula.** Renormalization absorbs the ultraviolet-divergent local part of bulk entanglement entropy into Newton's constant and higher-curvature couplings. It does not remove the finite, state-dependent bulk entropy. The semiclassical formula is schematically

$$S_A = \operatorname*{ext}_{\Gamma} \left[\frac{\mathrm{Ar}(\Gamma)}{4G_{\mathrm{ren}}} + S^{\mathrm{ren}}_{\mathrm{bulk}}(\Sigma_A) + S_{\mathrm{local\,ct}}(\Gamma) \right],$$

   with the appropriate higher-derivative terms. The Susskind–Uglum renormalization argument cannot be used to replace this expression by area alone for arbitrary excited states. Indeed, the cited FLM result identifies bulk entanglement as the one-loop correction.
   For a Weyl coherent state generated by a unitary strictly contained in the wedge algebra, the bulk entropy difference may vanish because the reduced state is unitarily conjugate to the vacuum state. If that special fact is what is needed here, it should be proved for the graviton algebra being used, including the constraint, edge, and corner sectors. It should not be inferred from a renormalization of G_N. The paper must either retain and evaluate $\Delta S^{\mathrm{ren}}_{\mathrm{bulk}}$ or formulate and prove a precise lemma showing that it vanishes for the stated class of states. Without this correction, equations (1.4), (1.6), and (6.16) conflate classical HRT area with semiclassical generalized entropy.

2. **The absence of a first-order HRT-area variation is not established.** For an extremal background surface Γ_0, displacement of the surface makes no contribution to the first variation, but the metric perturbation gives

$$\delta \mathrm{Ar}(\Gamma_0) = \frac{1}{2} \int_{\Gamma_0} \epsilon_{\Gamma_0}\, q^{ab} h_{ab}.$$

   Neither the condition $\delta\vartheta_{\pm}|_{\Gamma_0} = 0$ in equation (2.7) nor the linearized Raychaudhuri equation (2.13) sets this integral to zero. Equation (2.15) says that the first-order expansion is constant along a horizon generator; an independent corner or asymptotic condition is needed to set that constant, and even then the implication for the cut area must be shown. A gauge choice cannot remove a gauge-invariant first variation of the physical extremal area.

The reasoning from equation (2.9) to equation (2.12) is also incomplete. η^a is defined as the first derivative of the surface family at $\lambda = 0$, while the mean curvature is evaluated in the perturbed metric and assigned order λ^2. This does not determine the order of the actual surface displacement or control all second-order metric and embedding terms. The text even states that Γ_λ is displaced only at order λ^3, which does not follow from first-order extremality. The authors should solve the extremal-surface Jacobi equation in the chosen gauge and compute the area through second order, including the second-order metric backreaction. Until then, equation (2.16), on which equations (6.15)–(6.16) depend, is unproved. If a restricted trace-free radiative sector makes the linear variation vanish, that restriction and its boundary-CFT meaning must be stated explicitly and reconciled with the entanglement first law.

3. **The boost-flux formulas and the localization of the modular generator are inconsistent.** Equation (3.1) gives

$$F_\xi = \frac{1}{4\pi} \int dV\, d\Omega_2\, \delta\sigma_{AB}\delta\sigma^{AB},$$

whereas equation (6.8) implies on $\mathcal{H}_A^+$

$$F_\xi[\mathcal{H}_A^+] = \frac{1}{4\pi} \int dV\, d\Omega_2\, V\, \delta\sigma_{AB}\delta\sigma^{AB}.$$

The factor of V is essential for the Killing boost $\xi = V\partial_V$; these cannot be two definitions of the same Hamiltonian. For the complete bifurcate horizon, V also changes sign, whereas on the right half it is positive. The manuscript must distinguish the global boost generator, the one-sided weighted flux, and any unweighted affine-energy flux, with their signs, domains, Newton-constant normalization, and endpoint terms.

There is a related operator-algebraic problem in the paragraph preceding equation (4.2). The modular Liouvillean $-\log \Delta_{\omega_0}$ implements an automorphism on the GNS Hilbert space but is generally not an element affiliated with the local Type-III wedge algebra. A global boost charge is not made local merely because it is a classical phase-space observable. If its implementing unitaries belonged to the wedge algebra, the modular action would be inner, which is not the structure being invoked in the crossed-product construction. The paper should formulate the core using the modular automorphism as an external action and explain precisely which gravitational charge is affiliated with which algebra. This is not a minor wording issue, because equations (4.1)–(4.3) and the constraint $F_\xi - X = -C$ use these identifications.

4. **The crossed-product normalization and state require a precise construction.** Equation (4.1) sets the modular Hamiltonian equal to $2\pi F_\xi$, while the fixed-point discussion uses $F_\xi - X$ and the trace in equation (4.3) is weighted by e^X. Later, equation (6.7) contains $2\pi\langle X\rangle$. If X is the physical boost charge entering $F_\xi = X - C$, the canonical trace weight should be scaled consistently with the modular generator (equivalently one should introduce the dimensionless variable $x = 2\pi X$). As written, the group parameter, auxiliary coordinate, trace, and entropy formula use incompatible normalizations.

The core of a Type-III$_1$ factor is normally Type-II$_\infty$, with a semifinite trace rather than a finite normalized trace. A normal state has a Radon–Nikodym density that may be an unbounded, trace-measurable operator affiliated with the core; it need not be a bounded element of the algebra as asserted in equation (6.5). Its entropy also need not be finite. The authors should state the represented crossed product, dual action, trace convention, domain of the density, and hypotheses on f that guarantee normality and finite entropy. The distributional vectors $|X\rangle$ in equations (4.3) and (5.10) should be treated accordingly.

5. **No physical condition ties the auxiliary wave function to the second-order gravitational charge.** Equation (6.7) is an entropy for the product vector $|\omega_h\rangle \otimes |f\rangle$. The center and width of $f(X)$ are freely chosen. The subsequent statement that f may be peaked at the "classical value" of X does not construct a physical state satisfying the gravitational constraint, nor does it determine the second-order metric completion of the first-order shear data. Nevertheless, equation (6.9) replaces $\langle X\rangle - F_\xi$ by a corner Noether charge. This step needs a constraint equation imposed on states, together with boundary conditions that select $\delta^2 g$ and demonstrate that its asymptotic charge equals the chosen auxiliary expectation value.
   The distinction is especially important because the slowly varying condition used to obtain equation (6.7) makes f broad in X (its Fourier transform is narrow), while a sharply specified classical charge pulls in the opposite direction unless a controlled hierarchy of widths is supplied. The authors should construct a family of physical classical-quantum states, quantify the approximation error, and show that the charge/area relation survives that error at order λ^2. Otherwise the area term can be shifted simply by changing an auxiliary wave packet.

6. **The code-space compression is not automatically a von Neumann algebra, and the JLMS conclusion is largely assumed.** In general $P\widetilde{\mathfrak{A}}P$ is an operator system, not a subalgebra of $\widetilde{\mathfrak{A}}$: products satisfy
$$(P\tilde{a}P)(P\tilde{b}P) = P\tilde{a}P\tilde{b}P,$$
   which need not equal $P\tilde{a}\tilde{b}P$. One needs the code projection to reduce the boundary algebra, or a separately defined represented logical algebra. Thus the inclusion claimed before equation (5.4) and the Type-III factor conclusion after it do not follow as written. Standard entanglement wedge reconstruction is an operator-reconstruction statement on the code; it is weaker than an exact vacuum-preserving spatial isomorphism of the two local von Neumann algebras.
   If equation (5.3) is promoted to the explicit assumption of such a spatial $*$-isomorphism, then equations (5.5)–(5.14) mostly express the invariance of modular data under that isomorphism. This is a valid conditional observation, but it should not be advertised as a derivation of physical JLMS for the full boundary algebra. The authors must specify cyclicity and separating properties, normality, domains of the unbounded Tomita operators, complement reconstruction, and the perturbative order at which the isomorphism holds. They must also distinguish the background causal wedge used in Sections 2–4 from the state-dependent entanglement wedge that appears in JLMS. Relative entropy in a compressed code algebra is not automatically relative entropy in the original boundary subregion algebra.

7. **Appendix B does not prove purification independence.** Equation (B.3) asserts that two vector representatives that agree on $\mathfrak{A}$ also agree on $\mathfrak{A} \rtimes \mathbb{R}$. This is false in general: an extension contains implementing/auxiliary operators whose expectation values can depend on the commutant unitary relating two purifications. Equality of normal functionals on a smaller algebra gives no equality on a larger algebra without a specified common extension. A finite-dimensional analogue is immediate: two purifications with the same reduced density matrix are distinguished as soon as the observable algebra is enlarged to include the purifying system.
   Consequently equations (B.3)–(B.6) do not justify replacing the localized vector $U|\omega_0\rangle$ by the natural-cone vector $Uj_{\omega_0}(U)|\omega_0\rangle$ in the crossed-product entropy calculation. The authors should compute both induced states on a set of crossed-product generators or define a canonical extension from the algebraic state and prove that the two vectors realize it. Note also that equation (B.4) places the density in $\mathfrak{A}$ rather than in the crossed product. This appendix is central, because the relative-entropy orientation and the claimed support of the coherent excitation depend on the replacement it is meant to validate.

8. **The final HRT equality drops terms and overstates what equation (6.15) says.** Equation (6.7) contains $S(f)$ and charge/reference terms, and equation (6.15) retains unspecified state-independent

contributions. Equation (6.16) then identifies the absolute Type-II entropy with a vacuum-subtracted boundary entropy and a vacuum-subtracted area, with none of those terms. A Type-II$_\infty$ trace is itself normalized only up to scale, which shifts the entropy by a state-independent constant. The appropriate candidate statement would therefore be a difference such as

$$S_{\text{ext}}(\omega_h, f) - S_{\text{ext}}(\omega_0, f),$$

using the same trace and the same auxiliary state, followed by an explicit cancellation of every f-dependent and reference term. The manuscript does not perform this subtraction.

The authors should state and prove a precise theorem with the two sides, their algebras, states, trace convention, perturbative order, and error bound. The result may be a relation between the *state-dependent part* of a core entropy and a second-order Wald charge, which would still be useful. It is not currently an equality between the physical CFT vacuum-subtracted entropy and the HRT area as claimed in equation (6.16) and the abstract.

9. **The KMS and local-algebra inputs need correct hypotheses rather than heuristic proofs.** Appendix A infers weak clustering by asserting that the AdS vacuum is the unique vector invariant under the one-parameter Rindler boost. Uniqueness under the full AdS isometry group does not imply uniqueness under one subgroup, so $P_{\text{inv}} = |\omega_0\rangle\langle\omega_0|$ in equation (A.4) is an additional ergodicity assumption. The inverse temperature 2π is then cited rather than derived. It would be preferable to invoke the precise AdS Bisognano–Wichmann/KMS theorem and verify that its covariance, locality, passivity, and representation assumptions hold for the graviton horizon algebra.

   Likewise, Araki's result for ordinary free-field local algebras does not by itself establish that the shear algebra of constrained linearized gravity on a null horizon is a Type-III factor with Haag duality. Gauge degeneracies, zero modes, memory, corner observables, and the reflecting AdS boundary can generate centers or require an extended algebra. Since Type III, factorality, Haag duality, and cyclic/separating vacuum properties are used throughout Sections 3–6 and Appendix B, the paper should either prove them for the stated phase space or make them explicit assumptions and delimit the result accordingly.

Minor comments

1. The measure $d\Omega_2$ in equations (3.1), (3.5), (6.8), and the coherent one-point function is specific to a two-dimensional cut, while the paper claims all $d \geq 2$. It should be replaced by the invariant measure on the $(d-1)$-dimensional horizon cut, with all tensor contractions defined using its metric. Similarly, Section 5 calls $\partial\Sigma$ a two-surface.

2. The boundary of global AdS is the cylinder $\mathbb{R} \times S^{d-1}$ in the conformal frame used in Section 2. Calling its reference state the "Minkowski-invariant vacuum" requires an explicit conformal map to the Minkowski diamond; otherwise "CFT vacuum on the cylinder" is clearer.

3. The equality in equation (5.6) begins with a ket $|\widetilde{\gamma}_0\rangle$, which should presumably be $|\widetilde{\omega}_0\rangle$. The operators in that calculation should belong to the restricted algebra, not the full boundary algebra as the text states.

4. The right-hand side of equation (5.9) contains a tilded bulk extended algebra. Equation (5.10) also uses the bulk vacuum label in what is presented as the boundary trace. These notational errors obscure an already delicate identification and should be corrected.

5. The coherent displacement in equations (6.1)–(6.2) needs a derivation from the null symplectic form. In particular, the factor $16\pi^2$, the index contractions in $\delta\sigma(h)$, and the stated one-point function should be checked from the canonical commutator rather than asserted.

6. The statement below equation (6.7) that $S(\omega_0|\omega_h) = S(\omega_h|\omega_0)$ does not follow merely from choosing the natural-cone representative; Araki relative entropy is generally asymmetric and is independent of vector representative. If equality holds for this Gaussian displacement, the special calculation or exact cited proposition should be given.

7. Equation (2.8) uses an uppercase Σ where a summation symbol is intended. The homology footnote below equation (1.1) refers to P instead of A. "Hubeney" in the abstract should be "Hubeny." There are also many copy-editing errors, including "consitute," "vaccum," "now not," and "has been is computed." A careful language pass is needed after the substantive revision.

Recommendation

The operator-algebraic perspective is promising, and a carefully delimited theorem relating a crossed-product entropy difference to a second-order AdS Wald charge could be valuable. The present manuscript, however, does not prove its stated JLMS and HRT claims for the physical boundary subregion algebra. Because the failures concern the generalized-entropy formula, the perturbative area expansion, the definition of the modular flux, and the extension of states to the crossed product, they cannot be repaired by local edits. I recommend rejection, with resubmission encouraged only after these foundations have been rebuilt and the main theorem has been restated at the level actually supported by the analysis.