Referee Report

on "Spectator Axions in String Inflation and Primordial Black Holes"

Synopsis and overall assessment

The manuscript studies the two axionic partners of the Kähler moduli in a primordial-black-hole realization of Fibre Inflation. Section 2 starts from a three-modulus type-IIB large-volume compactification, freezes the volume and small-cycle sector, and obtains the three-field action in equation (19). The inflaton has the near-inflection-point potential in equation (21), while the two axions have field-dependent kinetic coefficients and the cosine potentials in equations (16)–(18). Section 3 reviews the single-field reference model and its ultra-slow-roll enhancement of the curvature power spectrum. Section 4 then identifies an "epsilon-floor" interval in which the axion kinetic energy exceeds that of the inflaton and the trajectory turns in field space. Section 5 interprets the turns through a two-field adiabatic–entropic decomposition and argues that a tachyonic entropic mode is subsequently transferred into curvature perturbations. Finally, Section 6 scans the axion decay constants and presents a three-field "Success" point whose peak power is enhanced from the nominal no-PBH baseline to $\mathcal{P}_{\mathcal{R}}^{\mathrm{peak}} = 3.5 \times 10^{-3}$.

The physical question is timely and worthwhile. Earlier work cited in the manuscript established both the single-field Fibre-Inflation PBH mechanism and geometrical effects of ultralight axions in Fibre Inflation, while the recent spectator-field studies cited in Section 1 showed that a subdominant field can substantially modify an ultra-slow-roll episode. Combining these ingredients in the intrinsically coupled string-derived field-space metric is a natural next step. The distinction between the two axions, and especially the possibility that one of them changes the small-scale spectrum without an equally large CMB-scale change, could be a useful result. The figures and the organization around the successive background phases make the proposed mechanism easy to follow.

At present, however, the central calculation is not sufficiently secure for publication. Equations (39)–(40) have the wrong sign for the axion potential force. Equation (67) gives an incorrect Christoffel symbol and omits another nonzero connection component. Equations (60) and (65) use opposite signs for the same field-space-curvature contribution. In addition, equation (42) calls a coordinate Hessian an axion mass even though the axion kinetic term is not canonical away from $\varphi = 0$, and equation (64) drops the metric and connection factors precisely where the entropic instability is explained. These are not peripheral typographical issues: they concern the background motion, the claimed tachyonic interval, and the conversion of entropy into curvature perturbations.

The phenomenological conclusion is also stated too strongly. The criterion $\mathcal{P}_{\mathcal{R}} > 10^{-3}$ is not a calculation of PBH formation or abundance, particularly in an ultra-slow-roll multifield system where the small-scale probability distribution can be strongly non-Gaussian. The three-field point in Section 6 is not analyzed with the two-entropy-mode formalism it requires, and the numerical setup is not specified sufficiently to reproduce the reported spectra and observables. Finally, the successful point requires a gauge-group rank of order 10^3 and a very small nonperturbative prefactor, but no compactification or scale-hierarchy check is provided that realizes these ingredients together.

I therefore recommend **major revision**. The paper could become a valuable contribution to JHEP or Physical Review D if the authors correct the covariant dynamics, demonstrate that the numerical results were generated with those corrected equations, provide a reproducible three-field analysis, and replace the threshold-based PBH language by a controlled abundance calculation or suitably restricted claims.

Logical structure, assumptions, and principal results

The effective theory rests on several linked assumptions. The compactification is taken to satisfy $\mathcal{V} \gg 1$ and $\tau_1, \tau_2 \gg \tau_3$; the complex-structure, dilaton, volume, blow-up, and θ_3 directions are integrated out; the remaining saxionic ratio is the inflaton; and the leading relevant shift-symmetry breaking for θ_1 and θ_2 is represented by one cosine per axion. The heavy fields are assumed to remain at their minima over a Planckian inflaton displacement. The axion energy is assumed subdominant, the initial angles are fixed to $0.1\pi f_i$, and the initial velocities are left implicit. On the cosmology side, the calculation assumes linear perturbation theory through the ultra-slow-roll and turn intervals, a Bunch–Davies initial state, 53 e-folds between the CMB pivot and the end of inflation, and a fixed mapping between a peak wavenumber and a PBH mass. PBH production is inferred from a universal power threshold rather than from a collapse calculation.

Within these assumptions, equations (13)–(20) define the proposed low-energy model. Equations (30)–(31) make the two potential amplitudes respond differently to the evolving inflaton, while equation (26) supplies opposite exponential kinetic factors. Equations (36)–(37) then split the Hubble slow-roll parameter into inflaton and axion pieces. The manuscript's central background claim is that the axionic piece can temporarily dominate the kinetic slow-roll parameter even while the axion energy remains subdominant, thereby extending the inflaton's stay near the plateau and causing two turns.

The perturbation argument is organized by equations (54)–(63). In the two-field reductions, a signed turn rate couples the curvature and entropy modes, and a negative effective entropy mass amplifies the latter. The second turn can then convert this amplified entropy mode into a persistent curvature enhancement. The scan in Figure 9 finds little effect at small decay constant for the fixed choices of W_0, $\mathcal{V}$, A_i, and initial angle, while the point in Tables 4–5 combines $f_1 = 0.1$ and $f_2 = 0.5$ with both axions to raise the small-scale peak. This is the main new result, but it depends on the corrected covariant dynamics and on a genuinely three-field perturbation calculation.

Major comments

1. **The axion equations of motion have the wrong potential-force sign, and the numerical implementation must be audited.** From the action in equation (26), the homogeneous equation for a coordinate χ with metric coefficient $G_{\chi\chi}$ is

   $$\ddot{\chi} + (3H + \partial_\varphi \ln G_{\chi\chi}\, \dot{\varphi})\, \dot{\chi} + G^{\chi\chi} V_{,\chi} = 0.$$

   Consequently, equations (39) and (40) should have $-e^{4\varphi/\sqrt{3}} V_{,\chi_1}$ and $-e^{-2\varphi/\sqrt{3}} V_{,\chi_2}$ on their right-hand sides, respectively. As printed, a positive initial displacement in the range used in Tables 3 and 4 accelerates away from, rather than toward, the cosine minimum.
   The authors should correct these equations and state explicitly which signs were used in the background solver and in the PyTransport model. If the printed equations entered any part of the calculation, all background and perturbation results in Figures 5–14 and Tables 5 and 7 must be recomputed. If the code used the correct covariant equations, a direct numerical cross-check should be included. The full initial data, including all field velocities and the criterion used to initialize them, are also required; field positions alone do not define the background solution.

2. **The field-space geometry in equations (60), (64)–(67) is internally inconsistent.** For either two-field reduction, the metric may be written

   $$ds^2 = d\varphi^2 + e^{-2b\varphi} d\chi^2, \qquad b = \frac{2}{\sqrt{3}} \text{ or } -\frac{1}{\sqrt{3}}.$$

It follows directly that

$$\Gamma^{\varphi}_{\chi\chi} = be^{-2b\varphi}, \qquad \Gamma^{\chi}_{\varphi\chi} = \Gamma^{\chi}_{\chi\varphi} = -b.$$

Equation (67) is therefore too large by a factor of two, and the statement that it is the only nonzero Christoffel symbol is false. Moreover, the unit entropy vector has $s^{\chi} = e^{b\varphi}$ when the background is directed along φ. Thus the approximation below equation (66) is missing the normalization factor $e^{2b\varphi}$ as well as the full covariant Hessian. The analogous omission appears in equation (64) after the trajectory rotates.

There is also a direct sign conflict. Substitution of equation (60) into equations (57) and (63) gives a curvature contribution $-\varepsilon H^2\mathbb{R}$, whereas equation (65) displays $+\varepsilon H^2\mathbb{R}$. The authors should fix a Riemann-tensor convention, derive the projection from equation (58), and use it consistently in the text, code, and Figure 6. Since negative field-space curvature is invoked as part of the instability mechanism, merely changing a sign in the prose is insufficient. A plot of the corrected covariant components, verified independently against the full perturbation mass matrix, is needed.

3. **Equation (42) is not the physical axion mass away from the vacuum.** The variables χ_i are canonically normalized only at $\varphi = 0$. Therefore $\Lambda_i(\varphi)/f_i^2$ is the coordinate curvature of the potential, not the instantaneous frequency of a canonically normalized axion fluctuation. Even at an axion extremum the potential part of that frequency contains

$$G^{\chi\chi}V_{,\chi\chi} = G^{\chi\chi}\frac{\Lambda(\varphi)}{f^2}\cos(\chi/f),$$

and the covariant entropic mass also contains the connection term $-G^{\chi\chi}\Gamma^{\varphi}_{\chi\chi}V_{,\varphi}$ and the curvature and turn-rate contributions. Equation (43) is unaffected at $\varphi = 0$, but the time-dependent interpretation of equation (42), Figure 4, and the comparisons with H in Section 6 are not.

The manuscript should distinguish the potential amplitude, coordinate Hessian, local canonical mass, and complete entropic mass. Figure 4 should be replaced or relabeled accordingly, and the numerical scan behind Figure 9 should be reinterpreted using covariant eigenfrequencies. The quoted boundaries near $f \simeq 0.1$ and $f_2 \simeq 0.3$ should not be presented as physical thresholds until this check has been performed.

4. **The numerical perturbation calculation is not reproducible, and the successful point requires a three-field analysis.** Equations (54)–(67) apply only after reducing to one axion, when there is one entropy direction. The principal "Success" model instead has three active fields and therefore two independent entropy modes. A single scalar $\mathcal{S}$ and one signed turn rate cannot describe its mode mixing. In addition, a quantum power spectrum is not obtained by evolving one classical solution for $\mathcal{R}_k$ and $\mathcal{S}_k$: the independent vacuum modes and their cross-correlations must be initialized and summed, or an equivalent covariance/transport system must be evolved.

The authors should give the complete three-field mass and turn matrices for the successful point, show the two entropy eigenmodes and their separate transfer into $\mathcal{R}$, and demonstrate that the result agrees between the transport implementation and a direct mode-function integration. They should also specify the starting value of $k/(aH)$, vacuum prescription, pivot normalization, all background velocities, solver tolerances, end surface, and definitions and evaluation scales of β_{iso} and $f_{\text{NL}}^{\text{equil}}$. Machine-readable spectra and the model/code input would be appropriate. In particular, Table 2 reports $f_{\text{NL}}^{\text{equil}} = 109$ while calling the reference model CMB-compatible; the scale and observational comparison of this number must be stated. The small-scale bispectrum and higher moments, not only a CMB-scale equilateral number, are essential for the PBH claim.

5. **A power-spectrum threshold does not establish PBH production or dark-matter abundance.** Section 3 adopts $\mathcal{P}_{\mathcal{R}} > 10^{-3}$ as a universal threshold, and Sections 6–7 use crossing of this line to state that PBHs are produced. Equation (25) only maps a wavenumber to a characteristic

horizon mass; it does not compute a collapse fraction. The abundance depends exponentially on the smoothed density distribution, the window and transfer functions, the collapse threshold and profile, critical collapse, the width of the spectrum, and the thermal and reheating history. Here it is especially important to include non-Gaussianity generated during ultra-slow roll and by entropy-to-curvature transfer. A peak of 3.5×10^{-3} may or may not yield an interesting abundance depending on these choices.

The revised paper should compute a PBH mass function and present the resulting fraction of dark matter, with a transparent range of collapse prescriptions and the relevant small-scale non-Gaussian statistics. Perturbative and, where relevant, stochastic control of the amplified modes should be checked. Until then, the defensible conclusion is that axions enhance the small-scale curvature power into a range potentially relevant for PBHs, not that the model produces PBHs or can explain all dark matter. The scalar-induced gravitational-wave spectrum should likewise be computed for the corrected spectrum if detectability is retained as a conclusion.

6. **Control of the string effective theory is not demonstrated for the parameter region that produces the claimed enhancement.** The passage from equations (5)–(7) to equations (16)–(18) suppresses the terms and backreaction that are neglected. For the successful point the paper estimates N_2 of order 10^3 and requires a very small prefactor A_2, while also allowing $a_2\tau_2 < 1$. The assertion in Section 6 that gaugino condensation then requires only $\tau_2 \gg 1$ is too quick. The condensation scale must lie below the Kaluza–Klein and string cutoffs, the four-dimensional gauge theory must have the assumed nearly pure matter content, threshold corrections must be controlled, and the large D7 stack must satisfy tadpole cancellation without invalidating the geometry or the moduli hierarchy. The existence of a large gauge group in a different F-theory construction does not by itself establish all these properties in the required K3-fibred model.

   At minimum, the authors should tabulate along the full successful trajectory H, the heavy-modulus masses, the Kaluza–Klein and string scales, the condensation scale, $\rho_{\rm ax}/\rho_{\rm inf}$, and the displacement of the integrated-out fields. They should show the hierarchy of the omitted terms in the scalar potential, including cross terms and other allowed nonperturbative contributions. If no globally consistent construction is available, the paper should present the successful point explicitly as an effective-theory target rather than a realized string model, and weaken the claim of controlled compatibility.

7. **The robustness and reduced-tuning claims require a defined parameter study.** Figure 9 varies a decay constant while holding $|W_0| = 1$, $\mathcal{V} = 10^3$, $A_1 = A_2 = 10^{-6}$, and the initial angles fixed. This one-dimensional slice cannot support statements about "most of parameter space" or generic axions. Conversely, the successful example changes V_0 relative to the no-PBH baseline and chooses two masses, two decay constants, two initial angles, and an exceptional nonperturbative prefactor. No measure of the alleged relaxation of the inflection-point tuning is given.

   The authors should define a tuning diagnostic, vary the inflaton and axion parameters jointly subject to CMB normalization and EFT constraints, and compare the allowed volume with a correspondingly reoptimized single-field model. Sensitivity to the initial angles and velocities is particularly important. A modest multidimensional scan would also show whether the suppression near $f_2 \simeq 0.2$ and enhancement above $f_2 \simeq 0.3$ are stable features or artifacts of the selected slice.

Minor comments and presentation

1. The final sentence of the three-phase discussion in Section 4 says $\varepsilon_\varphi \gg \varepsilon_\chi$ during the intermediate phase, while the definition, Figure 5 caption, and Section 5.2 require the reverse inequality.

2. Equation (48) should read $\delta\phi^I(x,t)$ on the right-hand side. The variables in equation (53) are gauge-invariant dimensionless curvature and entropy perturbations, not canonically normalized variables;

the canonical Mukhanov variables include the appropriate scale-factor normalization.

3. The Appendix A heading $C_4\,[\times 10^{-4}]$ in Table 6 is inconsistent with the $C_4 = 7.82 \times 10^{-7}$ and 7.00×10^{-7} entries in Table 1. This factor-of-10^3 ambiguity must be corrected because it prevents reproduction of the potential. The reference-model value of r is also given as 0.02 in Table 2 and 0.022 in Table 7.

4. The sentence following equation (11) contains the malformed expression for $\Delta\varphi$. In Section 1, "lighter than the inflation" should be "lighter than the inflaton." The doubled phrase "shown in black, shown in black" in Section 7 should also be removed.

5. The lower bounds asserted after equation (41), $\varepsilon_\varphi \gtrsim \varepsilon_\chi^2$ and $\varepsilon_\varphi \gtrsim (\rho_{\rm ax}/\rho_{\rm inf})^2$, require a derivation and qualifications concerning coefficients, cancellations, and the regime in which potential slow-roll expressions may be used during USR.

6. Table 7 should define whether the listed isocurvature fraction is evaluated at CMB horizon exit, at the end of inflation, or after a specified reheating history. This distinction matters when Section 7 expects the axions to decay after inflation.

7. The mass assignment in equation (25) should state its sensitivity to the reheating equation of state and duration. Fixing $N_{\rm CMB} = 53$ while also proposing axion-dominated reheating requires a consistency check of the pivot-to-mass mapping.

8. The "triplet" of gravitational-wave signatures in Section 7 is not a prediction of equation (26), which contains neither the axion–gauge nor the axion–gravity operators and specifies none of their coefficients. The text should say that such signals are possible in extensions, unless the relevant couplings and spectra are calculated.

9. Please state the metric signature and maintain a uniform convention for the sign of the kinetic Lagrangian between equations (4), (15), and the actions (19) and (26). Please also insert punctuation between the two mass formulae in equation (42).

Recommendation

The coupled-axion extension of the Fibre-Inflation PBH scenario is sufficiently interesting to merit reconsideration, and the qualitative mechanism could be important if it survives a covariant and reproducible calculation. The present version nevertheless contains errors in the central background and entropy-mass formulae and does not yet establish PBH formation or string-model control. I recommend **major revision**, with a new evaluation focused on corrected field-space dynamics, the complete three-field perturbation system, a quantitative PBH analysis, and an explicit accounting of the EFT hierarchies and tuning.