Referee Report

on "A Darboux-Theorem-Based Derivation of Geometric Structures in Various Non-Abelian Gauge Theories"

Synopsis and overall assessment

The manuscript applies a Faddeev–Jackiw–Dirac analysis to four systems on D-dimensional Minkowski spacetime. Section 2 reviews the finite-dimensional presymplectic Darboux argument. Section 3 then treats the Freedman–Townsend model, Yang–Mills theory, and non-Abelian BF theory. In each case the kinetic term is used to read off canonical pairs, the nondynamical components are identified as auxiliary fields or multipliers, and the resulting constraint algebra, reducibility hierarchy, Hamiltonian, gauge transformations, and local degree count are stated. Section 4 applies the same strategy to a nonlinear BF-type model whose target-space bivector $W_{ab}(\varphi)$ obeys the Poisson Jacobi identity. The central claims are that this nonlinear model has a first-class, field-dependent constraint algebra, no local degrees of freedom, an open Lagrangian gauge algebra, and an on-shell reducibility tower extending through stage $D-2$.

A compact comparison of these examples could be pedagogically useful. The canonical pairs and Gauss constraints in the Yang–Mills and linear BF subsections reproduce familiar structures, and the attempt to display the complete higher-stage hierarchy in arbitrary dimension is potentially useful. The presentation also emphasizes a sensible distinction between a closed Hamiltonian constraint algebra and an open Lagrangian gauge algebra.

In its present form, however, the work is not technically reliable enough for publication. The D-dimensional actions call $B^{\mu\nu}$ an ordinary two-form while using it in the component expression appropriate to the dual of a $(D-2)$-form; the corresponding issue affects H^{μ} in the nonlinear model. Unless these fields are explicitly defined as tensor densities obtained by Hodge dualization, the claimed BF actions are metric dependent and the all-dimensional reducibility interpretation does not follow. More seriously, the central nonlinear formulae contain undefined symbols, incompatible free indices, a missing operation sign, and inconsistent covariant derivatives. Equations (4.39), (4.41), (4.44), (4.51), and (4.57) cannot all be tensorial identities as printed. These are precisely the equations meant to establish closure and higher-stage reducibility.

The conceptual contribution is also unclear. In the examples, no nontrivial Darboux transformation is constructed: the displayed actions already expose their canonical kinetic terms. The standard Freedman–Townsend, Yang–Mills, and BF analyses are well known, while the relation of the nonlinear formulas to the Poisson-sigma, nonlinear-BF, BV, and AKSZ literature cited in the manuscript is not delineated. The claim that Faddeev–Jackiw methods have been confined mainly to finite-dimensional toy models and linear theories is too broad to serve as a novelty argument without a substantially more complete literature review.

I therefore cannot recommend publication in JHEP or Physical Review D. My recommendation is **rejection in the present form**. A substantially rewritten submission could be reconsidered if the authors first fix the form and density conventions, rederive and independently verify the nonlinear gauge and reducibility identities, supply the omitted regularity and counting arguments, and identify a genuinely new result relative to the existing Hamiltonian and BV analyses.

Logical structure, assumptions, and principal results

The general construction begins with the first-order action (2.1) and assumes that the presymplectic form $\mathrm{d}\alpha$ has constant rank, as stated in equation (2.3). Local Darboux coordinates split the variables into canonical pairs and null directions. Some null variables are eliminated algebraically through equation (2.5), while the rest become multipliers of the constraints in equation (2.6). The manuscript then assumes that, after any further restriction, the surviving constraints (2.7) and the Hamiltonian are first class, and invokes the Dirac conjecture to reconstruct gauge transformations. This step implicitly requires regular constraint surfaces, suitable boundary conditions, and projectability of the Lagrangian symmetries.

For the Freedman–Townsend model, equations (3.1)–(3.8) give the reduced phase space and flatness constraint. Equations (3.9)–(3.11) state a higher-stage covariant-derivative complex, equations (3.12)–(3.13) give the Hamiltonian, and equations (3.15)–(3.22) reconstruct an Abelian but on-shell reducible Lagrangian gauge symmetry. The claimed physical configuration-space dimension is N per spatial point. The Yang–Mills subsection uses the auxiliary field in equations (3.25)–(3.27), obtains the Gauss constraint (3.29), and recovers the usual irreducible gauge transformations (3.34)–(3.37) with $(D-2)N$ local polarizations. The BF subsection combines the Gauss and flatness constraints in equation (3.40), obtains the algebra (3.42), and claims a vanishing local degree count and the covariant transformations (3.48).

The nonlinear model is defined by equations (4.1)–(4.3). Its canonical pairs and three families of constraints are displayed in equations (4.4)–(4.8), while equations (4.10)–(4.13) give their field-dependent Poisson brackets. Equations (4.16)–(4.26) are intended to establish the Hamiltonian reducibility tower. Equations (4.35)–(4.38) then state the covariant Lagrangian symmetries; equations (4.39)–(4.45) purport to show that their commutator closes modulo Euler–Lagrange equations; and equations (4.46)–(4.59) purport to give the complete Lagrangian reducibility hierarchy. The principal nonlinear conclusions therefore depend directly on the formulae that contain the index and notation problems detailed below.

Major comments

1. **The form degree and density conventions are incompatible with the claimed D-dimensional BF interpretation.** In D dimensions, the standard topological pairing is

$$S_{\mathrm{BF}} = \int \langle B_{D-2} \wedge F_2 \rangle.$$

   It may be written as an integral of $B^{\mu\nu}F_{\mu\nu}$ only after B_{D-2} has been converted into an antisymmetric contravariant tensor density using the Levi-Civita symbol. Equations (3.1) and (3.38), however, describe $B^{\mu\nu}$ as the components of a two-form and freely raise and lower its indices with the Minkowski metric. If that statement is literal, $B^{\mu\nu}F_{\mu\nu}$ is a metric-dependent inner product of two two-forms, not the D-dimensional BF density. It agrees in form degree with ordinary BF theory only in four dimensions, after a missing dualization is supplied.
   The gauge law (3.18), with a three-index parameter, and its $(D-3)$-stage reducibility actually signal the dual-density interpretation. The same issue recurs in equation (4.1): a genuinely topological model requires B to be the dual of a $(D-2)$-form and H^μ to be the dual of a $(D-1)$-form, rather than the ordinary two-form and one-form language used later. This is not a cosmetic convention. It controls the measure, metric dependence, signs under index lowering, Euler–Lagrange derivatives, gauge-parameter degrees, and the degree count.
   The paper should be rewritten in differential-form notation or should define the dual tensor densities explicitly, including all Levi-Civita factors and their weights. The authors should then redo the space–time decompositions leading to equations (3.3), (3.39), and (4.4). Until that is done, the claims of topological invariance and arbitrary-D reducibility are not well posed.

2. **Section 2 does not establish the general Faddeev–Jackiw–Dirac procedure that the applications rely on.** The statement following equation (2.7) that the remaining constraints "can then be taken to be first class" is not true for a generic first-order system. Null vectors can generate new constraints, the pullback of the presymplectic form can still possess second-class directions, and the Faddeev–Jackiw iteration must continue until these possibilities are resolved. Eliminating only those constraints that happen to be explicitly solvable does not by itself prove that the remainder is first class. Likewise, the constant-rank hypothesis in equation (2.3) is never checked for the relevant restricted surfaces.
   In the four examples, the manuscript largely reads the canonical pairs and multipliers directly from actions already written in canonical form. No Darboux coordinate transformation, presymplectic kernel, rank calculation, or iteration is displayed. The authors should either supply these steps for each model, including the functional delta functions and boundary conditions, or narrow the methodological claim and title. They should also state the regularity assumptions under which the Dirac conjecture is being used. This is especially important for the nonlinear and reducible system, where field-dependent ranks or singular target-space Poisson strata can invalidate a uniform local count.

3. **The central on-shell closure equations of the nonlinear theory are not valid tensor equations as printed.** Several independent defects occur in equations (4.39)–(4.44). Equation (4.39) contains $\partial^{ae} B_{bc}$, although no object B_{bc} with two internal indices has been defined; the expected coefficient appears to involve a second derivative of W_{bc}. In equation (4.41), $\delta S/\delta B_{d\lambda\rho}$ is multiplied by a parameter carrying $\lambda\mu\nu$, leaving both ρ and ν unmatched. The last line has $A^{d\rho}\epsilon_{b\lambda\mu\nu}$, again leaving ρ and ν uncontracted, and there is no plus or minus sign between that term and the preceding $B_{b\lambda\mu}$ term. Equation (4.44) defines an object with free indices $\mu\nu$, but its first term contains $\xi'_{\lambda\mu}$, leaving λ free and omitting ν.
   These defects occur in the main advertised result, not in an ancillary formula. The authors should rederive the commutator of each transformation in equations (4.35)–(4.38), display a consistent set of composite parameters, and verify the result by direct substitution into the action. A useful check would be to show explicitly that the algebra reduces to the expected closed linear algebra when W_{ab} is linear in φ, and to the Abelian complex when W is constant or zero. Merely correcting isolated symbols will not be sufficient because the repeated free-index failures make the intended formula ambiguous.

4. **The higher-stage reducibility hierarchy also contains structural index errors and undefined operators.** The nonlinear covariant derivatives are defined with bars in equation (4.6), yet equations (4.19), (4.22), (4.27), and (4.28) use the unbarred D defined for the earlier Lie-algebra model. As written, those equations therefore insert the wrong connection. Equation (4.22) has a further counting error: its left-hand side ends at i_{p-1}, whereas the antisymmetrization on the right ends at i_p. Equation (4.51) has spatial indices i, j on its left but spacetime indices μ, ν on its right. Finally, equation (4.57) has p free μ indices but its displayed delta chain from β_3 through β_{p+1} supplies only $p-1$ slots; it also fails to antisymmetrize over all $p+2$ beta indices appearing on the left.
   The complete reducibility complex should be reconstructed with a uniform multi-index notation. For each stage, the paper should state the parameter space, the map to the previous stage, the allowed range of p or q, and whether the composition vanishes strongly, on the constraint surface, or only modulo Euler–Lagrange equations. Explicit $D = 3$ and $D = 4$ specializations would expose endpoint and stage-counting errors that are difficult to see in the current formulas.

5. **The constraint algebras and degree counts need an actual field-theoretic derivation.** Equations (3.7), (3.28), and (4.7) omit the spatial delta distribution, while equations such as (3.30), (3.42), and (4.10)–(4.13) are written as if they were finite-dimensional brackets. DeWitt notation is mentioned only intermittently. For derivative constraints, the unsmeared brackets contain derivatives

of delta functions, and integrations by parts can generate boundary terms and surface charges. The algebra should be presented in terms of smeared generators, with the assumed falloff or boundary conditions made explicit.

The physical degree counts are also asserted rather than demonstrated. For the Freedman–Townsend and BF models, the authors should give the alternating binomial count of independent flatness constraints after every reducibility stage. For the nonlinear model they should similarly count the combined G^a_{jk} and γ^j_a complex and specify the regularity assumptions on $W_{ab}(\varphi)$ and on the constraint map. Without such assumptions, an arbitrary Poisson tensor may have rank-changing strata, and a single degree count need not apply globally. The distinction between no *local* degrees of freedom and possible global or boundary degrees of freedom should also be maintained throughout.

6. **The paper conflates three different meanings of "on shell" and is internally contradictory about reducibility.** After equation (3.42), the flatness constraints are said to possess "off-shell reducibility" described by equations (3.19)–(3.22). Those equations are Lagrangian identities explicitly proportional to $\delta S^{\mathrm{FT}}/\delta B$, and hence describe on-shell reducibility; the conclusion correctly calls them on shell. At the Hamiltonian level, equation (3.9) is a strong Bianchi identity, whereas the higher compositions in equation (3.10) vanish only on the flatness constraint surface. These cases must not be grouped under one label.

   Similarly, equations (4.10)–(4.13) exhibit a first-class Hamiltonian algebra: the brackets close in the ideal generated by the constraints. Equations (4.39)–(4.42), by contrast, concern an open Lagrangian algebra closing modulo Euler–Lagrange equations. The introduction and conclusion should distinguish weak Hamiltonian closure, strong identities, constraint-surface reducibility, and Lagrangian on-shell closure. A table listing these properties for all four models would substantially improve the logical presentation.

7. **The novelty and relation to prior work are not established at the level expected by a high-energy theory journal.** The Yang–Mills calculation is textbook material, and the canonical structures of the Freedman–Townsend and BF theories are longstanding consequences of the works cited as Refs. [10] and [15]–[19]. The nonlinear model is placed near the Poisson-sigma and AKSZ literature in Refs. [20]–[24], but the manuscript does not say which of equations (4.10)–(4.59) are known, which are new, or how the proposed Faddeev–Jackiw derivation compares with the corresponding Dirac–Bergmann or BV master-action derivations. Reference [9], an earlier paper by the present authors, does not fill this gap.

   The authors should provide a focused literature comparison, identify a precise new theorem or construction, and demonstrate what the Darboux viewpoint proves that existing treatments do not. If the contribution is instead intended as a pedagogical rederivation, the claims and target venue should be adjusted. In either case, the unsupported assertion that the method has mainly been applied to toy models and linear field theories should be replaced by a documented survey.

Minor comments

1. State whether antisymmetrization brackets carry unit weight. The normalizations in equations (3.2), (3.9)–(3.11), and (4.16)–(4.26) cannot be checked unambiguously without this convention.

2. Define the invariant bilinear form used to raise and lower Lie-algebra indices. Semisimplicity gives a nondegenerate Killing form, but it does not justify silently replacing it by Kronecker deltas in every normalization.

3. In equation (3.13), use the Hamiltonian notation introduced in equation (3.12). In equation (3.16), $\bar{B}^{jk}_a$ is not defined; the multiplier in equation (3.6) is B^{jk}_a.

4. The parameter identifications preceding equations (3.18) and (3.48) omit the internal index on the barred three-form parameter. All components should be written with the same internal-index placement.

5. The sentence after equation (3.42) cites the Lagrangian relations (3.19)–(3.22) when discussing the Hamiltonian constraint subset. It should first cite equations (3.9)–(3.11) and then separately discuss the covariant Lagrangian lift.

6. Equation (4.6) ends with a dangling comma. The notation $\partial^{ab}W_{cd}$ and its higher analogues should be defined explicitly as successive derivatives with respect to the scalar coordinates.

7. Clarify whether the numbers N, $(D-2)N$, and zero refer to physical configuration-space degrees of freedom per point. Phase-space dimensions are twice the corresponding number.

8. The paper repeatedly says that results follow "immediately" or by a "straightforward computation" precisely where the nontrivial sign, index, and reducibility checks occur. Representative calculations should replace these assertions.

9. The bibliography and prose need a careful copy edit. Several entries lack final punctuation, capitalization is inconsistent, and "it results that" after equation (3.41) should be rewritten.

Recommendation

The manuscript has a potentially useful organizing idea, but its principal all-dimensional and nonlinear results are not presently verifiable, and the incremental contribution beyond established analyses is not identified. I recommend **rejection**. A future submission would need a ground-up revision rather than a typographical erratum: precise form/density conventions, a valid presymplectic derivation, corrected and independently checked closure and reducibility complexes, explicit degree counts and regularity assumptions, and a substantive comparison with the existing Hamiltonian and BV literature.