Referee Report

$w_{1+\infty}$ as the Frame Algebra of Kerr Soft Dressing

Synopsis

The manuscript proposes a physical interpretation of the celestial $w_{1+\infty}$ structure in terms of asymptotic frame data for scattering by a Kerr source. Its starting point is the Veneziano–Vilkovisky (VV) relation between the canonical Bondi frame and an intrinsic frame attached to the hard particles. Equations (1)–(6) review the leading construction: the boosted Schwarzschild shear is written as a trace-free sphere Hessian, and the canonical-to-intrinsic supertranslation is $T_{\mathrm{VV}} = 2G(p \cdot q)\log(p \cdot q)$ modulo the four ordinary translations.

The proposed extension introduces a rank-s parameter t, the soft charge (9), and a field-independent kernel whose even levels are electric and whose odd levels are magnetic, as defined in equations (13)–(15). The central identification (18) sets this kernel equal to the corresponding coefficient of the exponentiating Kerr source $S_\eta^{(0)} \exp(\eta\lambda a \cdot q)$. This selects one parity at each level. For aligned momentum and spin, $a \cdot q$ is affine in $p \cdot q$, and the resulting scalar inverse problem is integrated in closed form in equation (22) using the hyperbolic sine and cosine integrals. Equation (23) gives its level-by-level expansion, while equation (24) estimates the relative size of the first spin-dependent source.

The manuscript then combines the soft charge with a proposed hard charge and Ward identity, equations (25) and (38). It argues that the homogeneous part of the transformations has the polynomial Poisson algebra on T^*S^2 and that a local chiral restriction yields the classical $w_{1+\infty}$ bracket. At $s = 1$ the selected divergence-free vectors close into $\mathrm{SDiff}(S^2)$, equation (26). Finally, equation (29) is presented as the Kerr generalization of the VV frame dictionary, and equations (30)–(42) relate the construction to the spinning three-point amplitude, linearized Kerr multipoles, a Faddeev–Kulish-type displacement, and the soft-theorem Ward representation.

Assessment and recommendation

The proposed link among the VV frame shift, the exponentiated Kerr three-point amplitude, parity-alternating memory moments, and the celestial symbol algebra is imaginative and potentially useful. The aligned-spin calculation is the strongest part of the paper. In particular, direct differentiation of the functions in equation (22) does reproduce $\cosh(A + Bx)/x$ and $\sinh(A + Bx)/x$, and the coefficient formula (23) follows from expanding $(\alpha + \beta x)^s/x$ and integrating twice. The manuscript also states several important restrictions honestly: it treats a projected exponentiating contribution rather than the full soft theorem, fixes only one parity, does not preserve the Kerr locus under the displayed bracket, and has not constructed the higher-s homogeneous transformations.

Those restrictions, however, include precisely the steps needed for the main claim. For $s \geq 2$ the transformation law is assumed, only its principal symbol is retained, the hard flux is not derived, and the full charge algebra is not shown to close. There is also a concrete normalization inconsistency already at $s = 0$, and the state dependence and global solvability of the purported generators are unresolved. Thus the paper currently establishes a suggestive matching of source functions and a standard principal-symbol bracket, not a $w_{1+\infty}$ algebra of Hamiltonian Kerr frame transformations.

The relation to the literature should be described in these terms. The celestial $w_{1+\infty}$ current algebra, higher soft charges, VV supertranslation, generalized-BMS phase space, and exponentiated Kerr three-point amplitude are all taken from the cited literature. The new idea is their proposed identification through equation (18), together with the aligned-spin solution. This synthesis could support a valuable JHEP or Physical Review D article if the phase-space and Ward-charge construction were actually completed. A companion work cannot carry the proof of the central claim of the present paper.

I therefore recommend **rejection in the present form, with the possibility of a substantially revised new submission**. The required changes are structural rather than editorial: the authors must construct the charges and transformations whose algebra they advertise, repair the normalization, and demonstrate that the inverse problem has genuine global generators. If those points can be resolved, the aligned solution and the physical interpretation would merit reconsideration.

Major comments

1. **Equations (35)–(37) do not establish the advertised charge algebra.**

   Equation (35) postulates a homogeneous differential operator $\mathbb{L}_t^{(s)}$ for every s, and the text explicitly says that this operator is not constructed for $s \geq 2$. The argument then takes its principal symbol and uses the general identity that the principal symbol of a commutator of differential operators is the cotangent-bundle Poisson bracket. This proves equation (36) only for the highest-derivative symbol. It does not prove that the lower-derivative trace and curvature terms exist, that they preserve the chosen radiative phase space, or that their commutator is another transformation of the assumed form. Nor does it establish integrability of the Hamiltonians, the boundary functional in equation (37), or the Jacobi identity after field-dependent extensions are included.

   This is not a technical completion that can be deferred: equation (37) is called a central claim, and the title identifies its algebra with the physical frame algebra. The authors should construct $\mathbb{L}_t^{(s)}$ and the total charge at least through the first genuinely higher level, compute the complete brackets including all subleading derivative terms, and then give an all-level argument. Boundary conditions, degeneracies of the Ashtekar–Streubel form, integrability, and possible cocycles must be specified. If only the principal-symbol algebra is available, the title, abstract, and conclusions should instead state that the work finds a kinematic symbol-algebra analogy.

2. **The all-level Ward identity and hard charge are assumed rather than derived.**

   The familiar gravitational soft expansion has universal leading, subleading, and (with qualifications) sub-subleading information. Generic higher Taylor coefficients are not universal soft theorems. The manuscript tries to isolate an "exponentiating projection," but equation (42) is still written for arbitrary s as an unsmeared soft theorem, and equation (41) assumes a hard-state operator with no derivation. Likewise, equation (38) is checked only at $s = 0$ and $s = 1$. For higher s its two displayed flux terms are not shown to reproduce the required energy derivatives, spin operators, massive-particle terms, or theory-dependent contact contributions.

   The authors should state the precise class of amplitudes, perturbative order, and soft prescription for which equations (38), (41), and (42) are asserted. Starting from an established soft theorem or a covariant phase-space charge, they should derive all coefficients and boundary terms and demonstrate the LSZ action on massive spinning states. It is not enough to define the hard operator as whatever makes equation (25) true. If only the diagonal three-point exponent is meant, equation (42) must be replaced by a projected identity and distinguished explicitly from an all-order gravitational soft theorem.

3. **The leading normalization is internally inconsistent.**

   Equation (7) defines the helicity soft factor by $S_\eta^{(0)} = \mathcal{A}_{3,\eta}^{\mathrm{Schw}}/(p \cdot q)$ with $\mathcal{A}_{3,\eta}^{\mathrm{Schw}} = \kappa(p \cdot \varepsilon_\eta)^2/2$. Equation (8), in contrast, defines the sphere tensor $S_{AB}^{(0)}$ without the factor $\kappa/2$. The tensors in equation (13) are built from the helicity quantities of equation (11), so the right-hand side of equation (18) contains this extra factor. It therefore cannot reduce to equation (4), as claimed.

The mismatch is also visible dimensionally: the quantity in equation (7) is dimensionless, whereas the tensor in equation (8) has one power of mass; multiplication by G consequently gives different dimensions.

The paper needs a single stripped or unstripped amplitude convention. For example, if a stripped source $(2/\kappa)S^{(0)}_{\eta}$ is intended in equations (11)–(18), that definition must be made explicit and propagated through equations (19), (22)–(24), and (29)–(34). The authors should then show the $s = 0$ reduction with every factor of G, κ, ω, and the helicity-to-sphere projector displayed. At present the claimed anchor to the VV transformation is not normalized consistently.

4. **The inverse problem is solved only for a scalar, not for the rank-s generator.**

   Equations (14)–(15) define $\chi^{(s)}_t$ as an s-fold divergence (or its magnetic analogue) of a rank-s tensor t. Equations (22)–(23) solve for the scalar $\chi^{(s)}$, but no $t^{A_1\cdots A_s}$ is reconstructed. On a compact sphere this further inversion is neither unique nor automatic. A smooth rank-s trace-free tensor has restricted harmonic content, the multiple-divergence operator has a kernel, and low harmonics can obstruct its range. These issues become especially important because the displayed $\chi^{(s)}$ contains logarithmic functions with a full scalar harmonic expansion. Invoking unspecified trace descendants does not establish a global solution.

   For each parity, the authors should decompose the source into spin-weighted spherical harmonics, state the range and kernel of the map in equation (14), and construct a smooth global t or identify the modes that must be projected out. They should also fix its nonuniqueness and show that the same choice is compatible with the bracket. Without this step, the claimed all-orders "frame generators" in equation (23) are scalar potentials for a kernel, not demonstrated generators of the rank-s charge.

5. **The Kerr-selected parameters are state dependent and do not form the claimed sector.**

   Solving equation (18) makes t_s a function of the hard momentum and spin, $t_s = t_s[p, a]$. In a charge algebra, however, parameters are normally held fixed. If p and a are phase-space variables, their transformations add terms to the bracket; if they are fixed labels, charges tailored separately to different external states do not by themselves define one universal symmetry action. The manuscript also acknowledges after equation (26) that the finite-dimensional Kerr locus is not preserved by the bracket. It is therefore unclear in what sense that bracket is an algebra "of Kerr soft dressings," rather than the ambient algebra of arbitrary polynomial symbols.

   The authors should define the space on which the charges act, say whether (p, a) are dynamical or external, and compute the bracket with the resulting parameter dependence included. If the action leaves the Kerr locus, the physical meaning of the resulting non-Kerr frame data should be supplied. Alternatively, the claims should be restricted to an embedding of individual Kerr source profiles into a larger symbol space, not a closed Kerr frame algebra.

6. **The retarded-time transformation, frequency expansion, and memory observable are conflated.**

   Equation (16) gives an inhomogeneous transformation $u^s\mathcal{K}^{(s,0)}$, while equation (29) constructs a frame shift from $\omega^s\mathcal{K}^{(s,0)}$. The aligned solution assigns $[\chi^{(s)}] = L^{s+1}$ so that $\omega^s\chi^{(s)}$ has length, but then $u^s\mathcal{K}^{(s,0)}$ has a different dimension from the Bondi shear for $s > 0$ in the same convention. Equation (17) does not resolve this: moments of the news are distributional and acquire boundary terms, zero-mode subtractions, and regulator dependence. The later insertion of an arbitrary smearing function $\rho(u)$ changes the observable and prevents a unique identification with the unsmeared transformation.

   The manuscript should derive the Fourier conventions and dimensions from the radiative symplectic form, including all 2π factors and endpoint terms. It must distinguish (i) a large transformation of $C_{AB}(u,z)$, (ii) a coefficient in a small-ω source expansion, and (iii) a regulated memory moment. A concrete scattering waveform should be used to show that the proposed charge and regulated observable are finite and that the same kernel really connects these three notions.

7. **The coherent-state argument in equations (34), (39), and (40) requires a correct derivation.**

   Equation (39) integrates an on-shell massless phase-space measure together with $\delta(2p\cdot k)$. For a timelike future-directed p and a real positive-energy null k, $p\cdot k$ vanishes only at the soft endpoint. The formula as written therefore does not transparently produce a spatial Coulomb profile. Standard constructions of the stationary linearized field involve either a soft dressing kernel with its infrared denominator or an off-shell Fourier integral with a propagator. Those ingredients and their regulator are absent here.

   There is also an internal scope problem. Equation (34) says that Q_s^{Kerr} shifts the metric by the full linearized Kerr field, whereas the following paragraph says correctly that a fixed-s charge supplies only one term in the multipole tower. Equation (40) similarly assigns the full boosted Schwarzschild field to Q_s without explaining its s dependence. The authors should derive the displacement operator from a specified Faddeev–Kulish or classical-current construction, separate the generating charge from its fixed-level coefficients, and compute its metric commutator. The notation and claims in equations (34), (39), and (40) should then be made mutually consistent.

8. **The subleading and observable checks need to be demonstrated, not identified by parity alone.**

   Equation (20) fixes only the curl potential of a smooth generalized-BMS vector. The manuscript stresses that this is not the full residual transformation of the shear, but nevertheless identifies it with the spin memory datum and uses it as the first nontrivial member of the frame tower. An explicit comparison with the standard subleading soft charge is needed, including the homogeneous shear transformation, celestial-metric variation, hard angular-momentum action, and boundary conditions. This would provide a meaningful test of the construction before extrapolating to $s\geq 2$.

   Similarly, the percentages following equation (24) are ratios of local source terms at a selected angle, not predicted memory amplitudes. An observable depends on the scattering process, the frequency profile, the angular smearing, and the subtraction encoded by ρ. The paper should either perform such a calculation or relabel the estimates as illustrative ratios of generating-function coefficients. The zero at $\cos\theta = v$ is a zero of the aligned magnetic source, not by itself a detector-level null.

Minor comments

1. The label and surrounding discussion of equation (20) alternate between "divergence" and "curl." The equation fixes $\epsilon^{AB}D_A V_B = D^2\Psi$; the terminology should be uniform.

2. The symbol s changes role in the local basis below equation (26): $F_{m,s} = z^{m+s-1}p_z^{s-1}$ has polynomial degree $s-1$, whereas $\mathfrak{g}_s$ in equation (36) has degree s and the soft tower begins at $s=0$. An explicit dictionary among soft level, tensor rank, celestial spin, and the conventional $w_{1+\infty}$ label is needed.

3. The statement below equation (22) should take the joint soft limit $A, B \to 0$ at fixed A/B, not merely $B\to 0$. Translation constants that are discarded in this limit should be displayed, especially because the caption of Figure 1 imposes a definite normalization not written in equation (22).

4. The manuscript should specify the domain of $\log(p \cdot q)$ and distinguish the massive case, where $p \cdot q > 0$, from any massless limit. The reference scale removes only the stated translation when this domain and normalization are fixed.

5. Equation (9) writes a kernel that depends on C but does not state its functional differentiability or the falloffs needed for the u integral. The shifted news, ordinary news, and endpoint shear should be defined in one place.

6. The phrase "electric mass-multipole tower sits in Chi" and its magnetic counterpart in Shi is potentially misleading, because both functions occur in each line of equation (22) when $A \neq 0$. The parity statement applies to the coefficients after the joint expansion.

7. The companion paper repeatedly carries essential constructions but is not identified bibliographically where they are invoked. It should be cited explicitly, and the present article should separate results proved here from results conditional on that work.

8. The title and abstract should not say that the composition law "is" $w_{1+\infty}$ unless the full Hamiltonian transformations are shown to close. The result presently demonstrated is a local chiral reduction of the principal-symbol Poisson algebra.