Referee Report

Kerr Soft Dressing and the $w_{1+\infty}$ Frame Algebra at Null Infinity

Synopsis

The manuscript proposes an all-spin extension of the Veneziano–Vilkovisky (VV) relation between a canonical Bondi frame and an intrinsic frame adapted to a massive source. Its input is the part of a spinning three-point coupling that exponentiates in the classical Kerr limit. Writing the coefficient at order s as a parity-selected sphere tensor, the authors introduce a rank-s parameter $t^{A_1 \cdots A_s}$ and a linear soft charge

$$Q^{\text{soft}}_{s,\text{lin}}(t) = \int_{\mathcal{I}^+} du\, d^2z \sqrt{\gamma}\, u^s \mathcal{K}^{(s,0)}_{AB}[t] \widehat{N}^{AB}.$$

The kernel is an electric trace-free Hessian of a maximally longitudinal scalar for even s, and its magnetic dual for odd s, as in equations (3.5) and (3.9)–(3.10). The inverse equation sets this kernel equal to the real parity projection of the coefficient of $S^{(0)}_\eta \exp(\eta\lambda a \cdot q)$.

At $s = 0$ this construction reproduces the familiar VV solution $T_{\text{VV}} = 2G(p \cdot q) \log(p \cdot q)$ modulo translations. At $s = 1$ it fixes the curl, but not the divergence, of a sphere vector. For spin aligned with the momentum, the two invariants $x = p \cdot q$ and $y = a \cdot q$ obey an affine relation. The inverse problem then reduces to the two ordinary differential equations (7.12), whose electric and magnetic generating potentials are integrated in (7.13)–(7.14) in terms of the hyperbolic cosine and sine integrals. Their level expansion is displayed in equation (7.20).

The manuscript next defines hard Ward operators by differentiating $\omega \mathcal{M}_{n+1}(\omega q)$ at $\omega = 0$, proposes a flux representative, and combines the hard and soft pieces into total charges. A homogeneous differential action on the shear is then posited. Its principal symbols are polynomials on T^*S^2, whose canonical Poisson bracket gives a filtered tensor bracket. A local chiral reduction is identified with a $w_{1+\infty}$-type algebra. Finally, equations (7.8) and (7.24) interpret the whole tower as the difference between intrinsic and canonical Kerr frame data, while section 8 interprets the same kernels as regulated memory probes.

Assessment and recommendation

There is a potentially interesting synthesis here. The leading VV inverse problem is correctly reviewed, and the aligned calculation is a useful closed-form exercise: direct differentiation of equations (7.13)–(7.14) does give the two sources in equation (7.12), and the polynomial-plus-logarithm coefficient in equation (7.20) follows from integrating the binomial expansion twice. The parity bookkeeping is also clearly presented. These are concrete and reusable observations.

The central physical identification, however, is not established and in its present form conflicts with the asymptotic falloff of stationary Kerr multipoles. A mass or current multipole of order $\ell \geq 1$ resides below the Bondi shear order, whereas equations (7.8) and (B.101) put every member of the multipole tower into the shear itself. The $s = 1$ magnetic kernel moreover annihilates the $\ell = 1$ mode that represents the Kerr current dipole. These are not missing refinements: they directly contradict the advertised identification of the tower with Kerr frame dragging and higher stationary multipoles.

Several other indispensable steps are either assumptions or definitions. The all-s hard operator is defined from the desired soft coefficient rather than derived from a universal theorem; the higher-rank homogeneous transformation is not obtained from a gravitational phase space; and charge closure is inferred only after assuming transformation closure. The generic principal-symbol identity is valid, but it does not show that the proposed Hamiltonian charges realize that algebra. Worse, the explicit ordered operator offered as a completion has vanishing principal symbol for every $s > 0$ because of its alternating binomial signs.

The relation to prior work should therefore be stated more narrowly. The VV supertranslation, generalized-BMS Ward identities, higher soft charges, celestial $w_{1+\infty}$ currents, polynomial symbol brackets, and the exponentiated Kerr three-point coupling are established ingredients from the cited literature. The novelty is the proposed identification of one parity-selected source coefficient with a hierarchy of frame and memory kernels, together with the aligned inverse solution. Until the Bondi extraction and Hamiltonian realization are supplied, this is a conjectural matching of generating functions, not a demonstrated Kerr frame algebra.

I therefore recommend **rejection in the present form**. The problems below require a fundamental reformulation, rather than a conventional major revision. A new submission could become suitable for JHEP or Physical Review D if it identifies the correct asymptotic coefficients carrying the Kerr multipoles, constructs finite integrable charges and their transformations, and then proves the claimed algebra in that phase space. The aligned inverse calculation could form a valuable component of such a more limited paper.

Major comments

1. **The proposed Kerr-to-Bondi-shear map has the wrong radial order.**

   Equation (B.99) correctly defines the shear by $C_{AB} = \lim_{r\to\infty} r^{-1}h_{AB}$ after imposing Bondi gauge. A stationary source multipole of order ℓ has orthonormal spatial metric components of order $r^{-\ell-1}$. Converting both indices to angular coordinate components contributes r^2, so its contribution to h_{AB} is of order $r^{1-\ell}$. Only the monopole can therefore contribute to the $O(r)$ coefficient defining C_{AB}: the current dipole is $O(1)$, the mass quadrupole is $O(r^{-1})$, and higher multipoles fall still faster.

   This directly conflicts with equation (7.8), which identifies the full ω-weighted Kerr tower with one shear difference, and with equation (B.101), which calls the same relation an "analogous expectation." The radial system (B.111) is merely listed; it is not solved. A residual Bondi-gauge transformation can add pure-gauge boundary shear, but that does not make subleading physical multipole coefficients equal to C_{AB}.

   The authors should explicitly transform the linearized Kerr field to Bondi gauge through at least the spin dipole and mass quadrupole, display the full large-r expansion, and identify which Bondi coefficients contain each multipole. They must then formulate charges for those coefficients. If the claimed effect is instead a deliberately chosen pure-gauge shear encoding of subleading data, its residual vector, boundary conditions, and invertible map back to the physical multipoles must be exhibited. Without this calculation, the main dictionary is false at the level of radial power counting.

2. **The $s = 1$ kernel cannot represent the Kerr current dipole.**

   The manuscript repeatedly identifies equation (4.6), or equivalently (8.12)–(8.13), with frame dragging and the Kerr current dipole. Yet its own harmonic calculation in equation (B.127) states

   $$\widetilde{\mathcal{D}}_{AB} D^2 \Psi = \sum_{\ell \geq 2, m} -\ell(\ell+1)\Psi_{\ell m} \widetilde{\mathcal{D}}_{AB} Y_{\ell m},$$

   and immediately notes that $\widetilde{\mathcal{D}}_{AB}$ kills the $\ell = 1$ mode. The Kerr current dipole is precisely an $\ell = 1$ stationary current multipole. Thus the displayed kernel has no sensitivity to the datum it is said to measure.

   The angular function obtained by multiplying the boosted soft pole by $a \cdot q$ may contain $\ell \geq 2$ harmonics, but those harmonics are not thereby the Kerr $\ell = 1$ current multipole. The paper must decompose the actual linearized Kerr field and the proposed source into tensor harmonics and explain this discrepancy. The likely resolution is that angular momentum lives in a subleading Coulombic Bondi coefficient rather than in the shear kernel. In that case the $s = 1$ interpretation and all statements that the probe is the "memory image of the Kerr current dipole" must be replaced.

3. **The magnetic Ward kernel is not the inhomogeneous action of the generalized-BMS vector used in the algebra.**

   For a smooth vector Y^A, equation (4.5) gives the standard generalized-BMS inhomogeneous shear term $-2u(D_A D_B \alpha_Y)^{\mathrm{TF}}$, where $\alpha_Y = \frac{1}{2} D \cdot Y$. The minimal vector selected after equation (4.10) is divergence free, so this field-independent term vanishes. The magnetic shift in equation (4.6) therefore is not generated by the same generalized-BMS transformation. The manuscript acknowledges that (4.6) is a parity projection of a Ward kernel, but subsequently treats it as a frame transformation and combines it with the homogeneous part of (4.5). That combination has not been derived from one Hamiltonian vector field.

   The authors must specify an extended asymptotic phase space in which the magnetic shift is a well-defined canonical transformation, derive its full action on every boundary field (including the celestial metric when appropriate), and show integrability of its charge. It is not sufficient to borrow the homogeneous generalized-BMS action from one transformation and attach an unrelated coexact affine shift from another Ward projection. This also bears on smoothness: the explicit special solution (4.15) divides by $p \cdot \partial_z q$ and is undefined at its critical points, so it does not establish the claimed global smooth generator.

4. **The radiative phase space and the memory moments are not consistently defined.**

   Equation (3.2) treats C_{AB} and $\widehat{N}_{AB} = \partial_u C_{AB} - \tau_{AB}$ as independent canonical variables. On the ordinary Ashtekar–Streubel radiative phase space they are related by a differential constraint; the induced C–C bracket is nonlocal in retarded time. If an enlarged phase space is intended, its additional boundary variables, constraints, and Dirac reduction must be given. Otherwise the functional-derivative computations in section 6 do not establish the Hamiltonian flow of equation (3.3).

   There is also an elementary problem with equation (8.9). Since τ_{AB} is introduced as a u-independent Geroch tensor,

   $$\int du\, \widehat{N}_{AB} = \Delta C_{AB} - \tau_{AB} \int du,$$

   not ΔC_{AB}. A finite window or smooth cutoff leaves a regulator- dependent subtraction, which must be specified. More generally, the flow $\delta C_{AB} = u^s \mathcal{K}_{AB}$ grows polynomially for $s > 0$, changes the news, and lies outside the standard finite-flux falloffs. The regulated probe of equation (8.2) generates $\rho(u) u^s \mathcal{K}_{AB}$ plus endpoint terms, not the bare transformation used earlier. The authors should define one phase space and one boundary prescription, demonstrate finiteness and differentiability there, and repeat the bracket calculation with the resulting Dirac or covariant symplectic structure.

5. **The purported all-level Ward identity is definitional beyond the universal gravitational soft orders.**

   Equation (5.4) calls the sth derivative of $\omega \mathcal{M}_{n+1}(\omega q)$ an "sth soft theorem" for arbitrary s. Universal tree-level gravity controls the leading, subleading, and sub-subleading terms, subject to familiar qualifications; generic higher coefficients contain process- and theory-dependent contact information. Selecting the diagonal powers of a minimally coupled spin exponential does not turn those coefficients into universal symmetries of a general S-matrix.

   Indeed, equations (5.5)–(5.8) define the hard operator to be the right-hand side needed for the Ward equation, and the text correctly concedes that this is a representation statement. The flux in equation (5.9) is schematic: its conformal weights, improvements, normalizations, spin action, and massive $i^{\pm}$ contribution are not calculated. Consequently it is not shown to reproduce the proposed differential operator even at the first genuinely higher level.

The manuscript should separate the established soft theorems from a chosen Taylor coefficient of the Kerr three-point form factor. For each claimed charge it should state the theory, loop order, external states, and infrared prescription, and derive the hard action from a covariant phase-space flux or LSZ reduction. Explicit agreement at $s = 1$ and $s = 2$, including orbital and spin terms for a massive leg, is a minimum requirement before an all-s Ward symmetry can be claimed.

6. **The charge algebra is conditional, and the proposed ordered completion is algebraically incorrect.**

   Equation (6.9) begins with "if the affine transformations close." Equations (6.11)–(6.14) then use the Jacobi identity to infer the charge bracket after assuming precisely that closure. For $s \geq 2$, however, neither the homogeneous action nor the field-dependent kernel $\mathcal{K}^{(s,1)}$ has been derived. The identity (6.22) for principal symbols is standard and correct, but it fixes only the highest-derivative part of a hypothetical commutator; it does not construct the gravitational transformations, establish their boundary behavior, or prove charge integrability. Closing an abstract ordered-symbol algebra by definition does not supply those missing facts.

   There is also a decisive error in the advertised realization. Setting $s = 1$ in equation (6.59) gives

   $$\mathrm{Op}_{\nabla}(t)\psi = \tfrac{1}{2}\{t^A D_A \psi - D_A(t^A \psi)\} = -\tfrac{1}{2}(D_A t^A)\psi.$$

   The first-derivative term cancels, so the principal symbol is zero, not $t^A p_A$. More generally, the coefficient of $t^{A_1 \ldots A_s} D_{A_1} \cdots D_{A_s} \psi$ is $2^{-s} \sum_r \binom{s}{r}(-1)^r = 0$ for every $s > 0$. Therefore equation (6.60) does not have the asserted principal symbol, and the exact ordered bracket built from it cannot realize the preceding algebra. The authors must correct the ordering, specify all density and curvature terms rather than denote them by Ξ_t and $\mathcal{U}_t$, and verify the complete low-rank commutators on the shear before drawing any conclusion about a physical $w_{1+\infty}$ representation.

7. **The on-shell spin exponential has not been connected to the off-shell stationary Kerr field.**

   Equation (7.15) shows that the GOV contraction reduces to $i\eta a \cdot k$ only on complex three-point support $p \cdot k = 0$. At the generic real null direction used in the frame equation, the second, polarization-dependent term in (7.15) survives. The manuscript then adopts a different classical source exponential by definition and calls the three-point object its on-shell avatar. This is not a derivation of a stationary Coulombic field from radiative on-shell data.

   Equation (B.98) is only a schematic propagator formula, and section B.11 ends with a protocol for the Bondi gauge restoration rather than carrying it out. The required bridge must specify an off-shell conserved stress tensor, gauge completion, pole prescription, spin supplementary condition, and residual Bondi frame. The resulting metric should then be compared component by component with linearized Kerr and with the proposed asymptotic charge data. Until this is done, the assertions that the soft coherent profile is "precisely" linearized Kerr and that the same kernel occurs in all three representations remain conjectures.

8. **The complex-shift check contains explicit sign errors.**

   From equation (B.102), $\mathcal{Z}(\mathbf{x}) = M/\sqrt{(\mathbf{x} - i\mathbf{a})^2}$, the translation identity is

   $$\mathcal{Z} = M \exp(-i\mathbf{a} \cdot \boldsymbol{\nabla})\frac{1}{r} = M \sum_{\ell \geq 0} \frac{(-i)^\ell}{\ell!} a^L \partial_L \frac{1}{r}.$$

Equation (B.103) instead prints i^ℓ. Already at first order the direct expansion gives $+iM(\mathbf{a}\cdot\mathbf{n})/r^2$, whereas (B.103) gives the opposite sign. The error is also inconsistent with the Legendre expansion in equation (7.16).

At second order the real part of (B.102) is

$$\frac{M}{r} + \frac{M}{2r^3}(a^2 - 3(\mathbf{a}\cdot\mathbf{n})^2) + O(a^4) = \frac{M}{r} - \frac{Ma^2}{r^3}P_2(\cos\theta) + O(a^4),$$

which has the Kerr mass quadrupole $M_2 = -Ma^2$. Equation (B.107) places the opposite sign in front of this correction. These formulas are presented as normalization checks, so both signs and all dependent identifications must be recomputed before the Kerr matching can be trusted.

9. **A stationary frame relation is being conflated with a soft- frequency generating series.**

   The left-hand side of equation (7.8) is a geometric shear difference for a fixed stationary source, but the right-hand side depends on an arbitrary external graviton frequency ω. In the strict soft limit every $s > 0$ term vanishes. Conversely, at finite ω the series describes a frequency-space form factor, not a stationary Bondi frame. Retarded-time moments, Taylor coefficients at $\omega = 0$, and coefficients in the radial Kerr multipole expansion are three different kinds of data. Their common power of a generating parameter does not identify them.

   The authors should provide the Fourier or Mellin transform, endpoint prescription, and reconstruction formula that maps the soft derivatives into specific asymptotic metric coefficients. They should also explain why the answer is independent of the arbitrary probing frequency. Until then, the numerical finite-frequency estimates following equation (7.21) have no established relation to a Kerr frame shift or an observable waveform and should be removed.

10. **The inverse solution determines only a scalar projection, and the novelty claims require sharper comparison with existing work.**

    Equations (7.13)–(7.20) solve for the scalar $\chi_t^{(s)}$, not for the global tensor parameter $t^{A_1\cdots A_s}$, its trace descendants, or its homogeneous action. The multiple-divergence map has a kernel and a restricted harmonic image. A choice of preimage must be made and shown to be smooth, compatible with the selected parity, and stable under the proposed bracket. The manuscript itself notes in Remark 6.1 that the Kerr locus is not a subalgebra and that one potential does not determine the higher-rank tensor. Calling the aligned scalar an all-orders frame generator overstates the result.

    A revision should construct the preimage mode by mode and state precisely what quotient is used. It should then compare the resulting charges, recursion, phase space, and brackets directly with the cited higher-spin null-infinity constructions, rather than relying on broad thematic citations. The polynomial Poisson algebra and its chiral $w_{1+\infty}$ reduction are standard; the paper's distinct contribution can only be the Kerr-selected embedding and its physical interpretation. Since the present embedding is neither closed nor connected to the correct Bondi multipole coefficients, its novelty and significance for a high-energy theory journal are not yet demonstrated.

Minor comments

1. Equation (6.39) appears to subtract the tensor-index term twice. With $\mathcal{L}_Y$ denoting the standard Lie derivative of a contravariant rank-two tensor, the Poisson bracket is simply $\{F_Y, F_t\} = (\mathcal{L}_Y t)^{AB} p_A p_B$; the additional $-2t^{C(A}D_C Y^{B)}$ is inconsistent with equation (B.77).

2. The scalar bracket in equations (6.32) and (6.27) omits the conformal- density term that appears in the usual generalized-BMS action on a supertranslation, conventionally $Y^A D_A T - \frac{1}{2}(D\cdot Y)T$. The

parameter weights and the relation between the polynomial scalar and a BMS supertranslation should be fixed explicitly.

3. The distributions $\delta^{(s)}(\omega)$ in equations (3.39)–(3.44) are used on the half-line $\omega \geq 0$. The one-sided normalization, possible half-weight at the endpoint, integrations by parts, and boundary terms should be stated.

4. The symbols s and ℓ should not be interchanged implicitly. The order in the spin exponential need not equal the harmonic number of the boosted angular source, especially in the presence of the pole $1/(p \cdot q)$.

5. The conventions for W_η, $\mathcal{W}_\eta$, $a * k$, the bookkeeping variable λ, and the physical frequency ω change across sections 3 and 7. A single table of definitions and dimensions would make the sign and reality conditions checkable.

6. The manuscript is approximately ninety pages and repeats the same claim-and-caveat sequence many times. Several appendix subsections labelled as checks are proposals or future calculations, notably the Bondi matching in B.11. A substantially shorter version should distinguish proven statements, assumptions, definitions, and conjectures typographically and remove repetitions.

7. The title and abstract should not say that the frame algebra or the Kerr dictionary has been established unless the radial, phase-space, and closure problems above are resolved. "Parity-selected inverse kernels for the Kerr spin exponential" would describe the present mathematical result more accurately.