Referee Report

Vacuum fluctuations in Rainbow space-time: Study of Casimir effect

Synopsis

The manuscript studies a massless scalar field interacting with two semitransparent parallel plates in a rainbow-space-time background. The background is described by the energy-dependent metric

$$ds^2 = -\frac{dt^2}{f^2(E_o/E_P)} + \frac{d\boldsymbol{x}^2}{g^2(E_o/E_P)},$$

and the plates at $z = z_1, z_2$ are represented by delta-function potentials with strengths λ_1 and λ_2. Section 2 reviews the modified dispersion relation and three choices of rainbow functions. For the calculation, these are expanded as $f = 1 + \alpha E_o$ and $g = 1 + kE_o$, with the third choice restricted to the power $n = 1$.

Section 3 constructs the scalar Lagrangian, equation of motion, and stress tensor. Equations (3.8)–(3.14) rewrite the integrated energy density in terms of a reduced Green function. Section 4 then expands the deformed Green function as in equation (4.3), derives the inhomogeneous equation (4.12) for its first-order correction, and writes the corresponding resolvent convolution in equation (4.13). The manuscript asserts that this convolution vanishes and therefore concludes in equations (4.14)–(4.15) that the entire deformation is an overall multiplicative factor multiplying the Minkowski Green function.

Using that result, Section 5 imports the standard two-delta-plate Green function of equations (4.8)–(4.11), separates bulk, plate self-energy, and interaction terms, and obtains a Casimir energy proportional to $1 - 2kE_o$. Thus the first and third rainbow choices are claimed to decrease and increase the magnitude of the force, respectively, while the second choice is claimed to give no correction. In the perfectly reflecting limit, equations (5.13)–(5.15) give the usual scalar coefficients multiplied by this factor. Figures 1 and 2 plot these expressions, and the manuscript finally claims a bound of order 10^{-24} on the rainbow combination from a Casimir experiment.

Assessment, novelty, significance, and recommendation

The topic is potentially interesting. Casimir observables can provide a clean setting in which to test a proposed modified dispersion relation, and the use of semitransparent delta-function plates is more informative than starting directly from ideal Dirichlet boundary conditions. The manuscript is also organized around a recognizable logical chain: metric, action and stress tensor, Green function, interaction energy, and phenomenology. Its Minkowski-space ingredients are broadly consistent with the standard Green-function treatment cited in Refs. [20] and [40], and equation (5.13) does recover the familiar massless-scalar strong-coupling coefficient when the deformation is turned off.

However, the novel part of the paper is not correct. The resolvent convolution in equation (4.13) is generically nonzero. Dropping it invalidates equations (4.14)–(4.15), the overall factor in equation (5.4), all three model-specific conclusions in equations (5.10)–(5.15), and the plots. In addition, the calculation does not define whether E_o is an external probe energy or the energy of each fluctuating vacuum mode. These alternatives lead to different theories. If E_o is fixed, the displayed metric has constant coefficients and is related to Minkowski space by a coordinate rescaling; invariant plate separation, area, time, and energy must then be defined before any factor can be physical. If E_o labels the vacuum mode, the rainbow functions belong inside the frequency integral and cannot be pulled out as constants.

The claimed phenomenological bound is also untenable. Equation (5.15) predicts a fractional multiplicative correction, so an experiment with a one-percent relative uncertainty could constrain that fractional parameter only at roughly the percent level, not at 10^{-24}. The manuscript further mixes force, pressure, and force-gradient units and compares a parallel-plate scalar calculation with an electromagnetic measurement in a different geometry.

These issues remove the principal result rather than merely changing a coefficient. I therefore recommend **rejection in the present form**. The manuscript is not suitable for Physical Review D or JHEP without a new, internally consistent derivation. A substantially reworked paper could be considered as a new submission if it specifies the physical meaning of the rainbow metric for vacuum modes, derives the correct Green function and conserved energy, performs a regulator-consistent subtraction, and replaces the experimental analysis. The main points requiring resolution are listed below.

Major comments

1. **The convolution in equation (4.13) does not vanish.**

   Equation (4.12) is an inhomogeneous resolvent equation. Once g is the inverse of the operator on its left-hand side, its solution is precisely the convolution displayed in equation (4.13):

   $$g_1(z,z') = g(z,z') + \int d\bar{z}\, g(z,\bar{z}) \left[2\frac{\alpha-k}{\alpha+k}\omega^2 + \frac{k}{\alpha+k}V(\bar{z}) \right] g(\bar{z},z').$$

   There is no support property restricting the $\bar{z}$ integration to the interval between z and z'. The Green functions in equations (4.8)–(4.10) are nonzero throughout the relevant regions. Even with $V = 0$, the integral of the product of two free resolvents is nonzero (it is equivalently a derivative of the resolvent with respect to its spectral parameter). With the plates present, the potential term explicitly contains

   $$\lambda_1 g(z,z_1)g(z_1,z') + \lambda_2 g(z,z_2)g(z_2,z'),$$

   which is also generically nonzero.

   This problem affects each of the three choices. For the first choice $\alpha = k$, the frequency part vanishes but the plate-potential part remains. For the second choice $k = 0$, the frequency part remains. For the third choice $\alpha = 0$, both structures contribute. Thus equation (4.14) is not obtained in any of the plate configurations being studied, and equation (4.15) cannot be used in equation (5.3).

   The authors should solve the reduced equation directly or evaluate the convolution rather than discard it. In fact, before expanding, equations (3.6) and (4.2) imply, in the manuscript's conventions,

   $$-\left[\partial_z^2 + \frac{f^2}{g^2}\omega^2 - k_\perp^2 - \frac{1}{g}V(z) \right] \widehat{g}(z,z') = fg\,\delta(z-z').$$

   After Wick rotation the inverse decay length is $\sqrt{k_\perp^2 + (f^2/g^2)\zeta^2}$ and the delta-plate couplings become λ_i/g. Expanding this exact expression supplies an immediate check on every first-order term. All results following equation (4.13) must be recomputed from this corrected Green function.

2. **The role of the probe energy E_o is undefined for a vacuum fluctuation calculation.**

   Equations (2.1)–(2.9) describe a geometry probed by a particle of energy E_o, but the Casimir energy is a sum over vacuum modes of all frequencies and transverse momenta. The calculation treats E_o as one fixed external number while independently integrating over ω and $k_\perp$ in equations (3.12), (3.14), and (5.2)–(5.4). No physical prescription relates E_o to those modes, to the plates, or to the laboratory observer.

   If E_o is the energy of each virtual mode, then f and g are functions of the integration variables. The dispersion relation, density of states, Wick rotation, integration measure, and ultraviolet behavior all change, and the constants αE_o and kE_o cannot be extracted from the integrals. The resulting integrals also require a statement about the domain of the rainbow functions and the regulator near the Planck scale. If instead E_o is an external apparatus or observer scale, the authors must explain why the quantum field propagator is evaluated in that observer-selected metric and what observable is compared between different choices of E_o.

This ambiguity is particularly serious for the force. Equation (5.15) differentiates with respect to L while keeping E_o fixed. If the intended scale is a characteristic Casimir energy of order $1/L$, the rainbow factor itself is L dependent and its derivative gives additional terms. The paper must choose a prescription at the level of the theory, not after obtaining the Minkowski result, and demonstrate that it is compatible with the modified dispersion relation in equation (2.1).

3. **Equation (5.1) is not the conserved energy in the stated static metric, and coordinate quantities are confused with proper observables.**

   For
   $$g_{\mu\nu} = \mathrm{diag}(-f^{-2}, g^{-2}, g^{-2}, g^{-2}),$$
   the lapse is $N = 1/f$, the spatial volume element is $\sqrt{h} = g^{-3}$, and the future unit normal is $n^\mu = (f, 0, 0, 0)$. The conserved charge associated with the Killing vector $\xi^\mu = (1, 0, 0, 0)$ is therefore
   $$E_\xi = \int_\Sigma d^3x \sqrt{h}\, n^\mu T_{\mu\nu} \xi^\nu = \int d^3x\, \frac{f}{g^3} T_{00}.$$
   Equation (5.1) instead uses $\sqrt{-g}T_{00} = T_{00}/(fg^3)$, differing by f^2. It is neither the Killing energy nor the local comoving energy. This error alone changes the α dependence and is central to the surprising claim that the second rainbow choice, for which only f changes, produces no effect.

   Moreover, equation (2.6) implies a proper plate separation $L_{\mathrm{prop}} = L/g$ and proper area $A_{\mathrm{prop}} = A/g^2$, while proper time is $d\tau = dt/f$. Equations (5.2)–(5.15) never state whether L, A, and E are coordinate or proper quantities. For fixed constant f and g, the coordinate change $\tau = t/f$, $\boldsymbol{X} = \boldsymbol{x}/g$ makes the metric explicitly Minkowskian. A physical correction can then be claimed only after expressing the answer in invariantly measured separation, area, and energy and showing what remains beyond this coordinate rescaling. This check is especially important for the first choice $f = g$, where the metric is a constant overall rescaling. The authors should derive the Hamiltonian charge from the action, state the observer and normalization of time, and rewrite all results in proper laboratory quantities.

4. **The experimental bound following equation (5.15) is dimensionally, statistically, and geometrically unsupported.**

   According to equation (5.15), the two claimed fractional corrections are $\delta P/P_0 = -2a\varepsilon$ and $\delta P/P_0 = a\varepsilon$. If a measured observable agrees with the standard prediction to relative accuracy u, these formulas could at best imply $|a\varepsilon| \lesssim u/2$ or $|a\varepsilon| \lesssim u$, before systematic uncertainties. Taking the manuscript's stated one-percent accuracy would therefore give numbers of order 5×10^{-3} or 10^{-2}, not 10^{-24}. The paper supplies no calculation that bridges this difference of more than twenty orders of magnitude.

   The comparison also mixes different observables. Equation (5.15) is a force per area, with SI units of $\mathrm{N\,m^{-2}}$. The text calls $8 \times 10^{-4}\,\mathrm{N\,m^{-2}}$ a "force gradient," while a force gradient has units $\mathrm{N\,m^{-1}}$. Figures 1 and 2 label the same per-area expressions as force in N and energy in J rather than pressure in $\mathrm{N\,m^{-2}}$ and energy per area in $\mathrm{J\,m^{-2}}$. The necessary factor of $\hbar c$ for converting equations (5.13)–(5.15) from natural units is not shown.

   Finally, Ref. [35] includes an electromagnetic sphere–plate measurement over 0.1–0.9 μm, whereas the manuscript invokes scalar parallel plates at 10 μm. Geometry, electromagnetic polarizations, conductivity, roughness, temperature, and the measured quantity must be matched before a bound is quoted. The entire phenomenological paragraph, the abstract, and the conclusion should be withdrawn unless the authors provide a unit-complete prediction and a likelihood or conservative error analysis for an appropriate dataset.

5. **The stress-tensor and Green-function calculation lacks a consistent point-splitting and renormalization prescription.**

   Equations (3.10)–(3.13) take the coincidence limit of a singular two-point function without specifying a regulator or subtracting the no-plate vacuum. The spatial total derivative is then set to zero by the divergence theorem in passing from equation (3.13) to equation (3.14). With delta-function plates, the reduced Green function has derivative jumps at z_1 and z_2. Integrating the divergence piece by piece generates interface terms; whether these are assigned to surface energy, absorbed into plate self-energies, or cancel after a specified subtraction must be demonstrated rather than assumed.

   The same issue reappears in equations (5.6)–(5.9). The bulk term and the two single-plate energies are divergent, but no regulator, counterterm, or $L \to \infty$ subtraction is stated. The finite interaction energy can be defined cleanly, for example from the logarithm of the two-plate determinant after subtracting the isolated plates. The authors should keep the points separated until the subtraction is made, derive the surface contributions, and show that the force obtained from the renormalized interaction energy agrees with the discontinuity of the renormalized stress across a plate.

6. **The covariant starting point and perturbative control need to be stated more carefully.**

   Equation (3.1) gives a Lagrangian but does not explicitly write the action $S = \int d^4x\sqrt{-g}\,\mathcal{L}$. The determinant later appears in the Green-function source and in equation (5.1), so its placement is not merely a notation choice. The delta-potential normalization in equation (3.2), the canonical stress tensor in equation (3.7), the Green-function normalization in equation (4.2), and the conserved Hamiltonian should all be derived from one action with clearly identified coordinate and proper plate couplings.

   The ansatz (4.3) is also singular when $\alpha + k = 0$, even though the preceding formulation is advertised as general. A two-parameter expansion $\widehat{G} = G + \alpha E_o G_\alpha + k E_o G_k$ avoids artificial denominators and makes the three models transparent. The paper should state the conditions $|\alpha E_o| \ll 1$ and $|k E_o| \ll 1$, estimate the omitted terms, and explain why the values $a\varepsilon = 0.05$ and 0.1 used in the plots lie in a controlled regime. For the third rainbow function, all conclusions should explicitly be limited to $n = 1$; the general n introduced in equation (2.9) has not been treated.

7. **The relation to prior work and the claimed novelty must be reassessed after the calculation is corrected.**

   The standard semitransparent-plate Green functions and the Minkowski Casimir integrals are taken from the literature represented by Refs. [20] and [40]. The Introduction also surveys noncommutative-space-time, generalized- uncertainty-principle, and rainbow-Einstein-universe calculations. As written, the paper's new result is the overall rainbow multiplier and the experimental bound. Since both depend on the invalid equations (4.14)–(4.15) and on the energy definition in equation (5.1), the present manuscript does not yet establish a new physical effect.

   A revised work should compare its corrected spectral determinant or Green-function result with the closest rainbow-space-time Casimir literature, separate genuine modified-dispersion effects from constant coordinate rescalings, and identify which result is not already fixed by the known Minkowski delta-plate calculation. Any comparison with the noncommutative or generalized-uncertainty-principle papers should be made at fixed physical separation and with common definitions of energy and pressure; agreement of the sign of an unverified multiplicative factor is not by itself significant.

Minor comments

1. The symbol λ denotes a rainbow parameter while λ_1, λ_2 denote plate couplings. Since the strong-coupling limit is also written as $\lambda_1, \lambda_2 \to \infty$, distinct notation would substantially improve readability.

2. The manuscript alternates between E_o and E_0, for example in equations (3.2)–(3.3), and alternates between hatted coordinates, Minkowski coordinates, coordinate plate positions, and proper positions without a definition table. These conventions should be standardized.

3. The sentence introducing equation (5.13) calls the displayed quantity the Casimir force, although the equation is the energy per area. The force per area first appears in equation (5.15).

4. The phrase "commutative space-time" after equation (5.12) is not the limit defined in the manuscript; "undeformed Minkowski limit" would be more precise unless a noncommutative coordinate algebra is explicitly introduced.

5. In equations (4.8)–(4.10), hyperbolic functions should be typeset as operators, and the region conditions should say explicitly whether both z, z' lie in the indicated region. The derivation and jump conditions at each delta plate would be useful checks on these long formulas.

6. The plots should state the assumed plate area, whether $\hbar = c = 1$ has been restored, the scalar boundary condition, and the exact numerical inputs. At present their axis units do not match equations (5.14)–(5.15).

7. The abstract and conclusion repeatedly say that, for two choices, the absolute values are "decreasing or increasing." State separately which choice gives which sign and do not report the result until the Green-function calculation has been corrected.

8. A thorough copyedit is needed. Examples include "Wein's" rather than "Wien's," "upto," "the first term is a divergent," "error less than 1 percentage," inconsistent capitalization of rainbow space-time, and several subject–verb disagreements.

Conclusion

The manuscript addresses a reasonable question and uses a useful semitransparent-plate setup, but its central first-order Green-function solution is mathematically incorrect and its observable energy is not properly defined. The subsequent Casimir factors, plots, sign comparisons, and experimental limit therefore do not follow. A publishable treatment would require a new calculation beginning with a clearly specified action and meaning of E_o, followed by the exact or correctly perturbed reduced Green function, an invariant energy and distance prescription, a controlled renormalization, and a unit- and geometry-consistent experimental comparison.