Referee Report

"A regulated zero-temperature construction of the Fundamental Modular Region in pure Yang–Mills theory and QCD"

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Recommendation

I cannot recommend publication of the manuscript in its present form. The paper contains a potentially useful organizing idea: at fixed lattice spacing and finite volume, a Gibbs measure on gauge transformations certainly localizes on the absolute minima of the lattice Landau functional as the inverse copy temperature tends to infinity. This gives a clean way to relate a continuous finite-temperature orbit measure to absolute Landau gauge and to simulated-annealing or best-copy searches.

The central claims go substantially beyond that elementary localization statement, however, and the derivations do not presently support them. Most importantly, the paper uses three inequivalent objects as though they were one: a discrete sum over stationary Landau copies, a continuous Gibbs integral over all gauge transformations, and the Serreau–Tissier replicated construction. It also alternates between normalization on each gauge orbit and one normalization after integrating over all gauge fields. The latter does not merely choose one representative of every orbit; it reweights different physical orbits by their minimum Landau functional. In addition, the advertised $1/\beta$ coefficient omits terms from the cubic variation of the Landau functional and from the integration measure. These issues affect the definition of the proposed theory, its purported local representation, and its principal numerical diagnostic.

A major rewriting would therefore amount to a new technical analysis rather than an ordinary revision. I would encourage the author to choose one finite-regulator measure, prove its equivalence (if any) to the replicated theory, retain the orbitwise normalization throughout, and redo the saddle expansion and numerical tests. A reconstructed paper could be of interest to a journal such as *Physical Review D* or *JHEP*. The present manuscript is not yet suitable for either journal, and its length and technical scope are also not well matched to a Letter format.

Synopsis and logical structure

Sections 1 and 2 review the Gribov ambiguity in Landau gauge and define a Boltzmann-weighted average over stationary copies A^{U_i}. The absolute minima of the Landau functional define the Fundamental Modular Region Λ. Section 3 observes that, for the discrete average in equation (13), all copies above the minimum are exponentially suppressed. Equations (19)–(21) describe the limiting average over minimizing copies. The section then replaces the discrete picture by a continuous expansion near an isolated minimizer and states in equation (25) that the leading correction is $(2\beta)^{-1}\,\mathrm{Tr}(\mathcal{M}^{-1}O^{(2)})$.

Section 4 gives a finite-lattice prescription. Equation (28) averages over gauge transformations with a normalization that depends on the underlying lattice orbit, and equation (30) orders the limits as $\beta \to \infty$, $\zeta \to 0^+$, $V \to \infty$, and $a \to 0$. Section 5 claims that this finite-β prescription has a local replicated continuum representation, displayed in equations (35)–(39), and supplements it schematically with a refined Gribov–Zwanziger sector. Section 6 argues that a gauge-invariant horizon functional is constant along an orbit and therefore does not alter normalized copy weights. Section 7 proposes lattice tests involving gluon and ghost propagators and the low-lying Faddeev–Popov spectrum.

Section 8 presents a proof-of-concept calculation on twelve $SU(2)$ configurations on a 4^4 lattice at Wilson coupling $\beta_{\mathrm{W}} = 1.2$. It compares low-budget overrelaxation, simulated annealing, population annealing, a neural warm start, and isotropic or Faddeev–Popov-guided basin hops. The benchmark-best value is defined as the lowest value found by any included method. The appendices expand the localization argument, repeat the saddle calculation, give the replicated action, discuss the horizon factor, describe the algorithms, extend the prescription to QCD, and offer a Hamiltonian interpretation.

Assessment of correctness, novelty, significance, and clarity

The finite-regulator Laplace principle is correct in its limited form. On a finite lattice the gauge group is compact, the Landau functional is continuous, and a normalized positive Gibbs measure concentrates on its minimizing set. Population annealing is a natural numerical realization of that measure. The manuscript also correctly emphasizes that finding a true global minimum remains computationally difficult and that the small benchmark does not certify membership in the exact FMR.

The proposed advance would be significant if the author established a single regulated measure that (i) selects the FMR orbit by orbit, (ii) has the stated local replicated representation at finite β, (iii) possesses a controlled and correctly derived approach to the limit, and (iv) admits stable continuum and infinite-volume limits. The present text does not establish these points. Several displayed definitions are mutually inequivalent, and the leading asymptotic formula is incomplete. Consequently, the claims that the paper gives a local continuum regulator, a calculable finite-β correction, and a nonperturbative QCD generating functional are premature.

The novelty must also be separated from established ingredients. Zero-temperature localization of a continuous function on a compact space is the standard Laplace principle; best-copy and simulated-annealing approaches to absolute Landau gauge are already represented in the cited lattice literature; and the local replicated weighted gauges originate in the cited Serreau–Tissier work. A worthwhile new contribution could be a rigorous bridge among these ingredients, together with a verified spectral convergence diagnostic. At present that bridge is asserted rather than derived, and the diagnostic is not tested in the numerical section.

The paper is readable at the level of motivation, but its logical structure is obscured by repeated transitions between different ensembles and by a very long appendix that sometimes contradicts the main definition. The title and abstract describe a construction, while the results currently amount to a proposed prescription plus a small optimization benchmark.

Major comments

1. **The paper must retain orbitwise normalization; equation (I.2) defines a different theory.**

   The definition in equation (28) has the structurally correct form

$$\int \mathcal{D}U\, e^{-S[U]} \frac{\int \mathcal{D}g\, W[U,g] O[U^g] e^{-\beta \mathcal{F}_U[g]}}{\int \mathcal{D}g\, W[U,g] e^{-\beta \mathcal{F}_U[g]}}.$$

   The denominator is evaluated separately on every orbit. This is essential: gauge fixing should select or average representatives without modifying the Yang–Mills weight assigned to the orbit itself.

   The “complete” prescription in equation (I.2), and likewise the schematic combined functional in equation (D.10), instead places one denominator after the integration over U. The two expressions are not equivalent. At large β, the globally normalized version weights an orbit by

$$e^{-\beta \mathcal{F}_*[U]} \big(\det{}' \mathcal{M}_*[U]\big)^{-1/2}$$

up to prefactors and degeneracies. It therefore drives the full Yang–Mills ensemble toward orbits with the smallest value of $\mathcal{F}_*[U]$; it does not select the minimum independently on every orbit. The passage from equation (D.10) to equation (D.12) drops precisely this orbit-dependent factor and is invalid.

All later formulas should be derived from one orbitwise-normalized definition. Equation (I.2) should be removed or replaced by equation (28), including its orbit-dependent normalization. The horizon, QCD, generating-functional, and Hamiltonian discussions must then be checked without ever interchanging an orbitwise ratio with a ratio of two full path integrals.

2. **The discrete-copy ensemble, the continuous Gibbs measure, and the replicated gauge are inequivalent as written.**

   Equation (13) is a sum over stationary Landau copies U_i. Equation (50) is an integral over all gauge transformations before stationarity is imposed. Appendix E then introduces a third object: a finite list of copies produced by relaxation. These ensembles have different finite-β behavior. At fixed finite copy list the approach is exponential, as equation (E.8) states. For a continuous nondegenerate minimum it is an asymptotic series in $1/\beta$. A set of relaxed copies is also weighted by the basin distribution of the search algorithm unless this sampling measure is explicitly corrected.

   The determinant prescription is internally inconsistent as well. The weight in equation (14) contains an explicit factor $s(i) = \operatorname{sign}\det \mathcal{M}_i$, while equation (C.1), from which the local replica construction is said to follow, omits that factor. When $\zeta \to 0$ and the determinant is nonzero, equation (14) tends to $+1$, whereas the factor in equation (C.1) tends to $s(i)$. Thus the two formulas disagree already before any replica limit. Moreover, the determinant ratio is motivated at stationary copies through a Faddeev–Popov change of variables, but equation (28) uses it throughout the continuous gauge group without defining the corresponding off-stationary operator and measure.

   The author should decide whether the primary regulator is a positive Haar-measure Gibbs integral over g, a signed sum over stationary copies, or another explicitly defined measure. An exact finite-lattice identity connecting it to the local replicated action must be supplied, including the delta functional imposing stationarity, the Jacobian, the sign convention, and the ζ regulator. Without this derivation, equations (35)–(39) and Appendix C are not a local representation of the measure used in Sections 4 and 7.

3. **The claimed leading $1/\beta$ coefficient in equation (25) and equation (B.13) is incomplete.**

   The derivation keeps only the quadratic term in the Landau functional and the quadratic term in the observable. For a generic gauge-dependent observable, its first orbit derivative does not vanish at a minimum of the Landau functional. The cubic variation of the Landau functional then combines with that first derivative at the same order $1/\beta$ as the displayed Hessian term. In finite-dimensional notation, with $C = (\mathcal{F}_*^{(2)})^{-1}$ and local measure density ρ, the coefficient contains terms of the form

$$\frac{1}{\beta}\left[\frac{1}{2} O_{ij} C_{ij} - \frac{1}{2} O_i C_{ij} \mathcal{F}^{(3)}_{jkl} C_{kl} + O_i C_{ij} \partial_j \log \rho\right],$$

   in addition to any derivative of $\log W_\zeta$. Equation (B.20) includes one contribution from W_ζ but still omits the cubic-action and coordinate-measure terms. These terms are absent only under additional symmetry or stationarity assumptions on O that are neither stated nor true for a general gauge-dependent observable.

The error directly affects the principal claim that the correction is determined by $\mathcal{M}^{-1}O^{(2)}$ and the proposed comparison with low Faddeev–Popov eigenvalues. The author should redo the functional Laplace expansion through order $1/\beta$, including the Haar Jacobian, global-gauge zero modes, determinant weight, cubic vertex, and possible multiple saddles. The range of uniform validity must be stated. In particular, the expansion is not uniform as an eigenvalue approaches zero, so proximity to the FMR boundary cannot be inferred by simply allowing the displayed inverse to diverge.

4. **The treatment of degenerate minima and the geometry of Λ is not correct for the continuous measure.**

   Equation (19) gives an equal-weight average over a discrete set of minimizing copies. A continuous Gibbs integral does not generically produce equal weights. For isolated minima g_α it gives, at leading order,

   $$p_\alpha \;\propto\; W_\zeta[U, g_\alpha]\big(\det{}'\mathcal{M}_\alpha\big)^{-1/2},$$

   and a manifold of minima requires a Morse–Bott measure involving its induced volume and normal Hessian. Boundary identifications, stabilizers, and constant gauge transformations therefore cannot be summarized as an automatic uniform residual average.

   There is also a conflict between the definitions and the later boundary discussion. Equations (10), (11), and (43)–(45) use strict positivity $\mathcal{M} > 0$ and write $\Lambda \subset \Omega$. Absolute minima have only a nonnegative Hessian and may lie on the Gribov horizon, while the Faddeev–Popov operator also has trivial global-gauge zero modes unless those transformations are quotiented. The appropriate statement involves the closure of the first Gribov region, with conventions made explicit. Appendix A's concentration argument should also be corrected: the probability outside a neighborhood generally carries a polynomial Laplace prefactor, so the bound in equation (A.6) with a β-independent constant multiplying exactly $e^{-\beta\Delta_{\mathcal{N}}}$ is not justified as written. A slightly weaker exponential bound is sufficient, but it must be proved correctly.

5. **The regulator scaling, replica limit, and continuum prescription require a real renormalization analysis.**

   The lattice functional in equation (26) is dimensionless and extensive, whereas the continuum functional in equation (7) has nonzero mass dimension. The parameter β in equation (37) behaves as a gluon mass squared in four dimensions, while the numerical inverse temperature multiplies an unnormalized lattice sum. The manuscript gives no map among these quantities as a and V vary, and it does not specify the scaling of ζ. Consequently, a finite-β comparison between lattices and the claimed continuum local action are not defined quantitatively.

   The replicated action also needs its full normalization and superspace measure. The replica terms in equations (36) and (C.7) display no explicit β dependence even though they are supposed to encode the Boltzmann weight, while the singled replica in equation (37) does. The analytic continuation $n \to 0$, its relation to the orbit normalization, and the order of the n, β, ζ, volume, and continuum limits are asserted but not controlled. Simply writing the ordered limits in equation (30) does not establish that the subsequent limits exist or are universal for gauge-dependent correlators.

   The revision should derive the lattice-to-continuum normalization of β and ζ, state the relevant renormalized parameters, and explain what is held fixed before the zero-temperature limit. If existence of the infinite-volume or continuum limit remains conjectural, the result should be called a regulated finite-lattice definition rather than a continuum construction.

6. **The numerical calculation neither samples the claimed observables nor tests the spectral prediction.**

   The benchmark compares optimization procedures after local relaxation. It does not present $D_\beta(p)$ or $G_\beta(p)$ at several β, fit a $1/\beta$ coefficient, or correlate a measured convergence rate with Faddeev–Popov eigenvalues. Thus it does not test the prediction emphasized in equations (55), (E.14)–(E.17), and the abstract. The result that population annealing and collective proposals find a lower basin than four random starts on one small orbit is useful algorithmic information, but it is not evidence for the analytical coefficient or for a continuum FMR prescription.

   The statistical basis is also too limited for a journal claim: twelve 4^4 configurations at one coupling, no production thermalization or autocorrelation study, no independent validation ensemble, and no uncertainty estimates. The "benchmark-best" value is defined using the same methods being compared, so the hit rate is intrinsically relative. Computational budgets are not equalized. In Table 1, simulated annealing has a mean Landau residual 2.79×10^{-6} while several other methods reach 10^{-10}; objective gaps at a 10^{-7} tolerance are not comparable until stationarity criteria are matched.

   A convincing numerical section should provide the configurations or generation protocol, code, random seeds, annealing schedules, sweep counts, overrelaxation parameter, proposal amplitudes, eigenvalue solver details, stopping criteria, and per-configuration results. It should compare methods at matched cost and matched residual, use independent reference searches, and directly test the finite-β observable and spectral statements. Figures 1 and 2 should be accompanied by numerical data and uncertainty information.

7. **The horizon, QCD, and Hamiltonian extensions presently overstate what follows from orbitwise localization.**

   If A^h is genuinely gauge invariant, the cancellation of a factor $H(A^h)$ inside an orbitwise normalized average is immediate. This does not by itself prove that the schematic action in equation (40) is a complete local BRST-invariant theory or that $\gamma \neq 0$ is an exact characteristic-function restriction to Ω. The schematic terms S_{transv} and S_{ref}, the horizon condition fixing γ, and the required transformations should be stated or the discussion should be explicitly limited to a formal compatibility observation. The global-normalization error in equation (D.10) must not be used in this argument.

   The QCD extension is similarly conditional on first obtaining the gauge-field measure correctly. The formal FMR functional in equation (G.10), with a Faddeev–Popov determinant and an integral over Λ, is not derived from the continuous Gibbs integral once its saddle weights are included. Gauge invariance of the quark action only shows that quarks do not change the orbitwise minimizer; it does not repair the gauge-fixing measure. Calling this a definition of the nonperturbative QCD vacuum is therefore too strong at present.

   Finally, Appendix H minimizes a functional of the spatial field and imposes $\partial_i A_i = 0$. This is the Coulomb-gauge FMR on a time slice, not the four-dimensional Euclidean Landau-gauge prescription developed in the main text. The relation between the two must be explained rather than called equivalent. The statement that the vacuum functional is simply constant on every gauge orbit also requires qualifications concerning global transformations, topological sectors, and possible theta-vacuum phases.

8. **The scope and novelty should be rewritten around what is actually proved.**

   The manuscript cites the essential Gribov–Zwanziger, Serreau–Tissier, and lattice best-copy literature, but it does not make a sufficiently sharp comparison. The zero-temperature concentration statement is a standard consequence of compactness and Laplace asymptotics. The existing weighted gauges already supply local replicated models, and simulated annealing already implements low-temperature searches for absolute Landau gauge. The new content would have to lie in a correct equivalence among a chosen positive orbit measure, the replica theory, and an FMR limit, or in a quantitatively verified Faddeev–Popov spectral correction.

The author should identify precisely which theorem or construction is new relative to the cited work, state all assumptions, and distinguish a definition from proof of a continuum limit. If the equivalence cannot be established, a more modest paper on a positive finite-lattice Gibbs prescription and its numerical sampling could still be useful, but the local-field-theory, QCD-vacuum, and controlled-coefficient claims would need to be removed or substantially weakened.

Minor comments

1. The notation should distinguish the inverse copy temperature β from the Wilson coupling β_{W} consistently and should state the normalization of the lattice Landau functional used in every table and figure.
2. The continuum and lattice functionals use different signs and normalizations. A short derivation connecting equations (7) and (26) would prevent ambiguity about whether absolute Landau gauge is a minimum or maximum in each convention.
3. The sentence following equation (G.12) is unfinished: the section ends immediately after the condition for a gauge-invariant observable. The intended conclusion should be restored.
4. The organization paragraph at the end of Section 1 omits the word "Appendix" before several references and begins its final sentence with bare cross-references. It should be rewritten as ordinary prose.
5. The source contains repeated grammatical errors, including "attaines," "setted," and awkward uses of "is localized." A thorough language edit is required.
6. In equation (C.11), β is interpreted as a transverse mass squared. Its mass dimension and relation to the parameter appearing in the replica superfield action should be stated immediately.
7. The manuscript defines a proposition environment but never states a proposition. The compactness and localization result would benefit from being formulated as a precise proposition with hypotheses and a correct proof.
8. Figure 1 uses a logarithmic scale although several methods evidently have values at or below the plotted floor. The treatment of zero, negative, or clipped gaps should be described. The phrase "rough-orbit" in the title is not defined.
9. Figure 2 shows recovery fractions close to one but no uncertainties, and its denominator can become unstable when the four-start gap is very small. The exact definition and the underlying gaps should be tabulated.
10. Table 1 should report integer counts as well as rounded percentages for a twelve-configuration sample. It should also include uncertainty or variability across random seeds rather than one aggregate outcome.
11. The distinction between a minimum in the interior of Λ, a boundary minimum, a degenerate minimum, and a minimum with only the trivial global zero modes should be maintained throughout. The inverse $\mathcal{M}^{-1}$ is not defined uniformly across these cases.

12. The phrase “absolute Landau gauge/FMR” should not imply a unique representative without qualification. Boundary identifications and stabilizers are part of the standard geometry and are central to the measure question raised here.

Final assessment

The manuscript begins from a sound and useful observation: a positive Gibbs measure on the compact lattice gauge group has a zero-temperature limit supported on absolute minima of the lattice Landau functional. That observation provides a natural language for population annealing and best-copy gauge fixing. It could support a valuable finite-lattice study and, with considerably more work, perhaps a controlled bridge to continuum weighted gauges.

At present, however, the manuscript does not define one consistent bridge. The global normalization in equation (I.2) changes the physical orbit measure; the discrete, continuous, and replicated ensembles do not coincide; the determinant sign conventions conflict; and the principal $1/\beta$ coefficient omits terms of the same order. The small numerical example does not test that coefficient or the proposed propagator limits. The QCD and Hamiltonian sections then inherit these unresolved issues and extend the claims beyond what has been shown.

For a high-energy theory journal, these are foundational problems of correctness and scope. A new version should first establish a mathematically consistent finite-regulator measure and its asymptotics, then demonstrate any local representation and numerical diagnostic. Only after those steps would the broader claims about the FMR, QCD, and continuum gauge-fixed correlators be ready for evaluation.