Referee Report

Generalized Kazakov–Migdal Models on Graphs via Artin–Ihara L-function and Random Partitions

Recommendation: Major revision

Synopsis

The manuscript studies Kazakov–Migdal (KM) type gauge theories on finite graphs and organizes them using graph bundles and Artin–Ihara L-functions. For a cycle graph and fundamental matter, the unitary matrix integral is expanded in characters and rewritten as a sum over Young diagrams with Schur weight. The authors then analyze the large N_c limit, recover the known Gross–Witten–Wadia (GWW) transition, and interpret it in three related languages: contact of the limiting Young diagram with the row-number boundary, occupation of a zero-energy Bose level, and a free-fermion droplet connecting unitary-matrix eigenvalue density to Maya-diagram density. A further claim is that the strong/weak functional equation has a combinatorial realization as replacement of a Young diagram by its complement in a rectangle.

The paper has an appealing organizing idea. In particular, equations (3.20)–(3.23) give a useful bridge from the FKM cycle model to the Schur measure, and the comparison between the unitary-matrix solution and the large-N_c saddle for the non-colliding integers is potentially valuable. The manuscript is also generally well motivated and its figures convey the intended phase structure effectively.

At present, however, several central derivations are not correct as written. The finite-N_c conversion to non-colliding integers contains an off-by-one error; the purported complement map in Section 5 is not a map of the stated configuration space; the step that produces the two droplet boundaries adds a pole residue absent from the original discrete Schwinger–Dyson equation; and the quantum-curve equation is inconsistent by a factor of two with both its defining wavefunction and its claimed semiclassical limit. These matters affect two of the principal novelty claims in the abstract. I therefore cannot recommend publication in the present form, but I would be willing to reconsider a substantially revised manuscript.

Logical structure and principal results

The analysis rests on the following sequence.

- Section 2 defines the weighted graph data and identifies the Gaussian matter determinant with an Artin–Ihara L-function. The determinant formula (2.17), representation decomposition (2.20)–(2.23), and tuned mass (2.25) are the main ingredients.
- Section 3 specializes to a cycle graph. Character orthogonality turns the FKM integral into the Schur-measure sum (3.20), and the specialization of the Schur polynomial gives the random-partition expression (3.23).
- Section 4 takes $\gamma = N_f/N_c$ fixed at large N_c. The unitary-matrix density in (4.6) has a transition at $(q^*)^L = (2\gamma - 1)^{-1}$. Independently, the Young-diagram variables lead to the singular integral equation (4.43), whose proposed solutions (A.8) and (A.12) reproduce the same thermodynamics. The row constraint is then interpreted through the limiting profile and the occupation m_0 in (4.47).
- Sections 5 and 6 make the stronger conceptual claims. Equations (5.2)–(5.3) are meant to realize strong/weak duality by a diagram complement, while equations (6.12)–(6.20) and (6.24)–(6.29) are meant to derive the droplet and quantum spectral curve rather than assume them.

This is a coherent plan, but the last two steps require major repair.

Major comments

1. **The finite-N_c dimension formula and integer shifts are inconsistent.**

   With the manuscript's definition $n_i = \lambda_i - i + 1$ in (4.35), one has $n_j = -j + 1$ when $j > N_c$. Hence the cross term in the Weyl product is $n_i + j - 1$, not $n_i + j$. Equation (4.40) should therefore contain

   $$\prod_{i=1}^{N_c} \frac{(n_i + N_f - 1)!(N_c - i)!}{(n_i + N_c - 1)!(N_f - i)!}$$

   rather than the displayed factorial ratio. A one-line check is $N_c = 1$, $N_f = 2$, $R = (k)$: the exact dimension is $k + 1$, whereas (4.40) gives $k + 2$. Likewise, equation (6.4) states $h_i = n_i + N_c$, but its two definitions imply $h_i = n_i + N_c - 1$.

   These shifts are subleading in part of the large-N_c saddle, so they need not destroy every limiting formula, but equations (4.41)–(4.43), the finite-N_c Fourier transform in Section 6, and all constant shifts in the proposed quantum mechanics should be rederived from a single consistent convention. The revised paper should state explicitly which conclusions are exact at finite N_c and which survive only at leading order.

2. **Equations (5.2)–(5.3) do not establish a Young-diagram complement duality.**

   As printed, (5.2) is undefined: for $i = N_c$ it calls for n_0. Replacing the index by the likely N_c+1-i does not solve the problem. The shift then sends allowed integers $n_1 > \cdots > n_{N_c} \geq -N_c + 1$ outside this domain and generally makes the factorials in (5.3) have negative arguments. Moreover, the ensemble in (3.23) bounds the *height* of a diagram but not its width. A complement requires a specified finite rectangle, and no common rectangle containing all diagrams in this infinite sum is supplied.

   There is also an analytic issue. The character expansion and probability measure are derived for $|q| < 1$, whereas $q \mapsto q^{-1}$ leaves that convergence domain. For $t = q^L$, the original unitary integral directly gives a functional relation of the form

   $$Z(t^{-1}) = t^{2N_c N_f} Z(t)$$

   after analytic continuation, using the determinant identity on the unit circle. This does not by itself provide a termwise bijection of the convergent random-partition series. The authors should either construct a valid configuration-space map, specifying the rectangle, prefactor, domain, and representation shifts, or recast the claim as an analytic functional equation without a diagram-by-diagram complement interpretation. Until this is done, the conclusion in Section 7 that the complement interpretation has been proved is not supported.

3. **The pole prescription in (6.14)–(6.17) changes the finite-N_c Schwinger–Dyson equation.**

   The discrete interaction term in (6.12) is $\frac{1}{2N_c}\sum_{j\neq i} \cot[(\theta_i - \theta_j)/2]$; it explicitly omits the self pole. Its continuum limit is the symmetric principal value. Replacing that sum by a one-sided contour and retaining either residue adds the self contribution $\pm\pi i\rho(\theta_i)$ in (6.15). The two one-sided boundary values have the principal value as their average, so the residue is not an ambiguity of the original real discrete sum. A choice of one side requires an additional analytic resolvent or complex-saddle argument.

   This matters because the added term is precisely what produces the two branches in (6.19) and hence the droplet boundary (6.20). The claim in the paragraph following (6.20) that the droplet has been obtained directly, without assuming its structure, is therefore not yet demonstrated. The authors should derive the appropriate resolvent discontinuity from the finite-N_c integral, explain how the ordered h_i are paired with the saddle variables θ_i, and justify the removal of the expectation brackets and large-N_c factorization. An alternative would be to treat (6.19) as an independently known collective-field relation, clearly separating that input from what is derived here.

4. **The proposed quantum curve has an explicit factor-of-two inconsistency.**

   Direct differentiation of the wavefunction defined in (6.9) gives an interaction term

$$\frac{1}{2N_c}\sum_{j\neq i}\cot\frac{\theta_i-\theta_j}{2}.$$

   Consequently, even if a one-sided continuum prescription is adopted, the corresponding terms are one half of the principal-value integral and $\pm\pi i\rho$, not the full principal-value integral and $\pm 2\pi i\rho$ displayed in (6.27). The mismatch is visible again in (6.29): its two branches differ by $2\pi\rho$, whereas the alleged semiclassical curve (6.19) differs by $\pi\rho$. Thus (6.27) does not follow from (6.9), and (6.28)–(6.29) cannot have (6.17) as their stated semiclassical limit.

   This is central to the claimed bridge to the quantum spectral curves of references [12,13]. The operator, its ordering, its domain, and the semiclassical scaling should be recomputed. The revised manuscript should also distinguish an exact annihilating operator at finite N_c from an operator obtained only after inserting a saddle-point density.

5. **The Bose–Einstein-condensation interpretation needs a defined thermodynamic limit and more cautious claims.**

   Equation (4.47) shows that $m_0 = N_c - \ell(R)$ is extensive below the GWW point, which is a useful combinatorial order parameter. It does not by itself establish Bose–Einstein condensation in a physical ensemble, because the weight $(\dim_{N_f} R)^2$ is a collective degeneracy rather than a product of independent level weights. The comparison model in (4.48) also does not support the sentence that follows it without a specified scaling limit. At fixed $q = e^{-\beta\varepsilon/L} < 1$, the expected excited occupation

$$\sum_{j\geq 1}\frac{1}{e^{\beta\varepsilon j}-1}$$

   is finite and independent of N_c, so the canonical system has $m_0/N_c \to 1$ as $N_c \to \infty$. An infrared divergence arises only in a limit in which the level spacing is simultaneously taken to zero.

   The paper should define the thermodynamic scaling, ensemble, chemical potential, and condensate criterion being used. It should then explain precisely how the weighted FKM partition differs from the ideal Bose gas and how this relates to references [30,31]. Without that analysis, "BEC-like occupation of the zero row" is defensible, while the stronger assertion of an intimate physical equivalence is premature.

6. **The exact saddle analysis contains contradictory phase inequalities and an incorrect integral identity.**

   From (A.11), $a = -2\gamma q^L/(1+q^L)$. In the saturated phase $q \leq q^*$, one has $a \geq -1$, with equality at the transition, not $a \leq -1$ as stated below (A.11). Likewise, the lower endpoint in (A.13) satisfies $a > -1$ for $q > q^*$, contrary to the sentence below (A.14). These are not merely stylistic inequalities: they describe whether the support touches the hard wall.

   In addition, the second identity in (A.26) is missing $d\theta$ and its prefactor should scale as $1/\sqrt{c_2^2-1}$, not $1/(c_2^2-1)$. The large-c_1 limit immediately reveals this: the integral behaves as $\log c_1/\sqrt{c_2^2-1}$. The subsequent expression appears to use the square-root version, but every appendix step should be checked and the agreement with (4.14) re-established after correction.

7. **The novelty relative to the cited literature should be delineated more sharply.**

   The GWW solution and its universality in this model are attributed to the authors' earlier work [2–7]; the droplet relation is associated with [10,11]; and the Schur-measure quantum spectral curve is associated with [12,13]. The present paper's potentially new content is the synthesis through (3.20)–(3.23), the hard-row-wall interpretation, a proposed direct derivation of the droplet discontinuity, and the proposed complement realization of the functional equation. Since the last two claims currently have the problems above, the Introduction and Conclusion should state, result by result, what is newly proved, what is rederived, and what is an interpretation of known formulas. This is important for assessing significance in a journal such as JHEP or PRD.

Minor comments

1. The definition preceding (2.5) says that a primitive cycle does not satisfy $C \neq B^r$. It should say that it is not of the form $C = B^r$ for $r \geq 2$.

2. In (2.17), the degree matrix in the $n_V d_R$-dimensional determinant should be written as $D \otimes I_{d_R}$. As printed, $D - (1-u)I_{n_V d_R}$ has incompatible dimensions.

3. Under (2.19), a cycle holonomy is conjugated by the group element at its base point; it is not itself invariant. The determinant in (2.14), and hence the L-function, is invariant because it is a class function.

4. The first equality in (3.7) assumes real q and a unitary representation. If complex q is retained after the earlier condition $|q| < 1$, the conjugate factor contains $\bar{q}$. The domain and reality assumptions should be stated consistently.

5. In Section 3, the gauge-fixing sentence should say that all edge matrices *except one* are set to the identity. Equation (2.26) should use S_R^Γ rather than an undefined S.

6. In (6.6), the power z_i^n should be z_i^k. In the first line of (6.12), h_i inside the sum over j should be h_j; the exponent in the following line also acquires an unexplained extra factor of i multiplying the potential. These errors obscure an already delicate derivation.

7. Equation (6.10) should specify the domain of the Fourier indices $\vec{h}$, the Weyl-chamber restriction, and any $N_c!$ normalization. Plain Parseval summation is over all Fourier modes, whereas Young diagrams correspond to ordered, nonnegative, distinct modes.

8. In (4.39) and the sentence following (4.43), $\rho(x)$ should be $\tilde{\rho}(x)$. "Poisonized" in Section 4.2 should be "Poissonized," "number of the ground states" should be "ground-state occupation," and "Young tableau" in the caption of Figure 2 should be "Young diagram." In Section 7, "expected to merge" should be "expected to emerge."

Publication recommendation

The manuscript contains a promising synthesis of graph zeta functions, unitary matrix models, and random partitions. If the Schur-measure reformulation is made conventionally consistent, the saddle analysis is corrected, and the status of the duality and spectral-curve claims is resolved, the work could become suitable for a high-energy theory journal. The present problems, however, concern the central derivations rather than presentation alone. I therefore recommend **major revision** and a new round of review before any publication decision.