Referee Report

Quantum Trigonometric Spin Ruijsenaars–Schneider Models from K-theoretic Coulomb Branches

Recommendation: Major revision. The manuscript should not be accepted until the algebraic inconsistencies below are corrected and the central commutativity claims are proved.

Synopsis and overall assessment

The manuscript proposes an algebraic quantization of the trigonometric spin Ruijsenaars–Schneider model using the quantized K-theoretic Coulomb branch of the four-dimensional $\mathcal{N} = 2$ necklace quiver with ℓ rank-N gauge nodes. Section 2 presents abelianized scalar variables Q_i^α, q-shift operators P_i^α, and minuscule monopole operators $u_i^{\alpha\pm}$. Section 3 packages these operators into two families of $N \times N$ matrices $L^{\alpha\pm}$ and states quadratic exchange relations involving an ordinary R-matrix, a dynamical R-matrix, and a dynamical twist. Multiplying the site operators around the necklace is claimed to reduce the exchange algebra to that of the spinless model.

Section 4 uses this reduction to introduce trace Hamiltonians $H^\pm[n]$, generalized Macdonald operators $S^\pm[n]$, a Newton-type determinant identity, and a quantum spectral curve. Section 5 identifies proposed Drinfeld–Jimbo generators of a horizontal quantum loop algebra, writes the lowest RTT modes, and relates $H^\pm[1]$ to the first nontrivial coefficient of the quantum determinant. Section 6 then defines quantum spin vectors and covectors, gives quadratic exchange rules, and writes Heisenberg equations for the physical variables.

This is an interesting program. A correct bridge between the Coulomb-branch difference-operator representation, quantum-toroidal symmetry, and the spin Ruijsenaars–Schneider dynamics would be valuable to the integrable-systems and gauge-theory communities. The attempt to make both the Hamiltonians and the physical spin algebra explicit is more useful than a purely structural existence statement. The manuscript is also reasonably concise and places the construction in the context of the Braverman–Finkelberg–Nakajima framework, earlier spinless trace formulas, the Krichever–Zabrodin model, and the Uglov–Lamers–Pasquier–Serban line of work.

In its present form, however, several formulas fail direct consistency checks. Equation (1.4) is not the $H^-[1]$ that follows from equation (4.2). More seriously, the $N = 1$ specialization of the defining formulas contradicts the normalization in equation (4.3) and exposes a sign conflict in equation (4.4). The proposed Drinfeld–Jimbo generators also have the opposite Cartan weights from equations (5.5) and (5.6), given the basic commutation rule (2.1). These are not matters of convention unless several connected definitions are changed together. In addition, the displayed L-operator relations do not establish mixed-sign commutativity, the full RTT representation is not constructed, and the stated algebra of physical spin variables is not closed on those variables.

The central ideas may survive after a systematic repair, but the current version does not yet establish the advertised quantum integrable model. A substantially revised paper could be suitable for JHEP, or for a comparable mathematical-physics venue, if it supplies complete algebraic checks and clearly separates proved statements from conjectures. The connection to high-energy theory is meaningful through line defects and Coulomb branches, but the gauge-theory interpretation should remain calibrated to what is actually constructed.

Logical structure and principal assumptions

The main argument appears to rely on the following chain.

- Equation (2.1) defines the localized q-difference-operator algebra. Equations (2.2)–(2.8) then give monopole operators and their exchange factors. This requires generic, nonzero deformation and mass parameters, invertibility of the displayed denominators, and an extension containing the required square roots.
- Equations (3.1)–(3.8) are intended to provide a consistent RLL algebra at one site. The twist relation (3.8) is then used to telescope the site-dependent data and obtain the total relation (3.10).
- The spinless trace theorem is applied to (3.10), producing the Hamiltonians (4.1). Equations (4.2)–(4.4) are intended to identify the same operators with generalized Macdonald operators and with coefficients of a normal-ordered determinant.
- Equations (5.2)–(5.12) specify the quantum-loop relations. The difference operators (5.18)–(5.20) are claimed to realize those relations, and the low RTT modes (5.22)–(5.23) are used in the quantum determinant formula (5.24), leading to equation (5.25).
- Equations (6.1)–(6.4) change from quiver variables to spin variables. Equations (6.5)–(6.13) are intended to define the reduced physical spin algebra, from which the equations of motion (6.15)–(6.17) follow.

Every later interpretation depends on the normalization and closure of these steps. The following points therefore need to be addressed before the main claims can be evaluated as established.

Major comments

1. **The introductory Hamiltonian (1.4) disagrees with the manuscript's own generalized Macdonald operator.**

   Setting $n = 1$ in the second line of equation (4.2) gives

   $$S^-[1] = \sum_{i_0,\ldots,i_{\ell-1}} \prod_{\alpha=0}^{\ell-1} \prod_{k_\alpha \neq i_\alpha} \frac{Q^{\alpha+1}_{i_{\alpha+1}} - \mu^\alpha Q^\alpha_{k_\alpha}}{Q^\alpha_{i_\alpha} - Q^\alpha_{k_\alpha}} \prod_{\alpha=0}^{\ell-1} (P^\alpha_{i_\alpha})^{-1}.$$

   Equation (1.4), which is presented as proportional to $H^-[1]$, omits every factor μ^α and uses $P^\alpha_{i_\alpha}$ rather than its inverse. Neither discrepancy can be absorbed into the indicated overall proportionality. This formula should be corrected, and the authors should check every later identification that uses it, including the interpretation as a Wilson–'t Hooft line.

2. **Equations (3.1)–(4.4) fail a decisive $N = 1$ normalization and sign check.**

   For $N = 1$, equations (2.2), (2.3), (3.1), and (3.2) reduce directly to

   $$L^{\alpha+}_{11} = q^{1/2} \left(\frac{Q^\alpha_1}{Q^{\alpha+1}_1}\right)^{1/2} P^\alpha_1, \qquad L^{\alpha-}_{11} = -q^{-1/2} \left(\frac{Q^{\alpha+1}_1}{Q^\alpha_1}\right)^{1/2} (P^{\alpha+1}_1)^{-1}.$$

   The square roots telescope around the necklace. Since all auxiliary matrices are one-dimensional, equation (4.1) then gives

   $$H^+[n] = q^{n\ell/2} \big(S^+[1]\big)^n, \qquad H^-[n] = (-1)^{n\ell} q^{-n\ell/2} \big(S^-[1]\big)^n,$$

   with $S^\pm[k] = 0$ for $k > 1$. By contrast, the determinant in equation (4.3) becomes $(S^\pm[1])^n$ and therefore predicts $H^\pm[n] = q^{\pm 1/2} (S^\pm[1])^n$. The powers of q disagree for generic n and ℓ. Moreover, the minus sign in $L^{\alpha-}$ implies that equation (4.4) has an extra factor $(-1)^\ell$ in its $S^-[1]$ coefficient as the definitions are currently printed.

This check uses no limiting argument and should be reproduced explicitly in the revision. It suggests at least two separate problems: the denominator sign in equation (3.2), and an ℓ- and n-dependent normalization missing from equation (4.3). The authors should first fix a single convention for $L^{\alpha\pm}$ and then rederive equations (4.1)–(4.4), rather than repairing individual prefactors. Checks at $N = 1$ for arbitrary ℓ and at $\ell = 1$ for small N should be included.

3. **The proposed Drinfeld–Jimbo realization has the wrong Cartan weights as printed.**

   Equation (2.1) states $P_i^\alpha Q_i^\alpha = qQ_i^\alpha P_i^\alpha$. For $\ell > 1$, equation (5.18) contains $(Q_i^\alpha/Q_i^{\alpha-1})^{1/2}$ in t^α, while every term in E^α in equation (5.19) contains P_i^α. Consequently the displayed definitions imply

$$E^\alpha t^\alpha = q^{1/2} t^\alpha E^\alpha, \qquad F^\alpha t^\alpha = q^{-1/2} t^\alpha F^\alpha.$$

   Equations (5.5) and (5.6) state precisely the opposite factors. The same reversal occurs for the neighboring Cartan generator. Thus the claim below equation (5.20) that the proposed generators satisfy equations (5.2)–(5.12) is false with the printed conventions.

   This mismatch affects the identification of the horizontal quantum loop algebra, the passage to RTT generators, and the quantum-determinant formula (5.25). Possible repairs include inverting the t^α, exchanging the roles of E and F, or reversing the stated weight convention, but each option changes other formulas. The complete Chevalley and Serre relations should be verified after the convention is fixed. A short explicit check for $N = 1$ and $\ell = 3$ would make the orientation unambiguous.

4. **The commuting family is not established by the relations that are displayed.**

   Equations (3.1), (3.2), and (3.10) give separate same-sign exchange relations for L^+ and L^-. They may justify commutativity within the $+$ family and within the $-$ family by the cited spinless trace argument, once the normalization problem is fixed. They do not imply $[H^+[m], H^-[n]] = 0$. In particular, the first relation in equation (4.6) requires $[H^+[1], H^-[1]] = 0$, because the stated energy and momentum are the sum and difference of these two operators.

   The revision should provide either a mixed L^+–L^- exchange relation with the corresponding trace proof, or a direct calculation in the difference-operator representation. It should also state exactly which set is claimed to be a maximal commuting family. The counting argument below equation (5.25) is explicitly called naive and is not a substitute: truncation can create relations in the image of a Bethe subalgebra, and algebraic independence of quantum Hamiltonians requires a precise meaning. At minimum, independence of their classical symbols on a dense open set should be proved if Liouville integrability is claimed.

5. **Section 5 does not yet construct the full RTT representation used in the conclusion.**

   The Drinfeld–Jimbo presentation could in principle define the desired homomorphism, but its proposed realization presently fails the weight test above. Independently, equations (5.22) and (5.23) give only the zeroth and first modes of $T^\pm(u)$, and the manuscript says only that equation (5.15) is checked to first order in the spectral parameters. This is insufficient to claim a representation of the full RTT algebra or to identify a full horizontal Bethe subalgebra in the image.

   Please distinguish three statements: a homomorphism from the Drinfeld–Jimbo presentation; formulas for the first RTT modes under a known isomorphism; and a representation of all RTT modes. Prove whichever of these is needed for equation (5.25), including the ordering and spectral shifts in equation (5.24). The manuscript should also distinguish the central quantum determinant from a general Bethe subalgebra. The conjecture in Section 7 that all $H^\pm[n]$ are central is interesting, but verification for unspecified "first few" values should be documented and cannot be used as an established ingredient earlier in the paper.

6. **The purported algebra of physical spin variables is not closed, and its derivation is too compressed.**

   Section 6 restricts a^α, c^α to $\alpha \neq 0$, but equation (6.9) contains $\widetilde{L}^{\text{tot}-}$ and the sum $\sum_{\rho=0}^{\alpha-1} q^\rho a^\rho c^\rho$. Thus the right-hand side still contains the excluded zero-spin variables. Equations (6.3) and (6.4) do not by themselves give a closed presentation in $Q_i^0, P_i^0, a_i^a, c_i^a$ with $a = 1, \ldots, \ell - 1$. No quantum reduction, constraint ideal, or explicit elimination of c^0 is supplied.

   The authors should state the quantum analogue of the classical symmetry reduction, show that its constraints are compatible with ordering, and express every right-hand side solely in physical generators. The passage from equations (6.11)–(6.12), whose R-matrix acts on particle indices, to the spin-index exchange relation (6.13) should be derived in components. Likewise, at least one of equations (6.16) or (6.17) should be derived in enough detail to check the ordering. Finally, the $q \to 1$ limit should be compared explicitly with the reduced Krichever–Zabrodin brackets and equations, including all normalization factors.

7. **The domain of the algebra and the status of the physical quantization need a precise statement.**

   The formulas use square roots of ratios of Q variables, inverses of $Q_i^\alpha - Q_j^\alpha$, and several mass-dependent denominators. The paper should specify the localized coefficient ring, the extension in which the half-powers live, whether q is formal or a generic complex number, and which root-of-unity and singular-mass loci are excluded. The geometric description near equation (1.3) calls multiplication by q a rotation; this requires $|q| = 1$, whereas the algebraic construction is normally used for generic q. The relation between the physical Omega-background regime and algebraic continuation should be stated.

   Section 7 correctly notes that Hermitian representations remain open. Accordingly, the present result is an algebraic deformation quantization, not yet a self-adjoint quantum-mechanical model with a specified Hilbert space or spectrum. The abstract and conclusion should maintain that distinction. This does not diminish the algebraic result, but it is important for the significance claimed in a high-energy theory journal.

Minor comments and presentation

1. In equations (3.10) and (4.1), $q^\mp$ should presumably be $q^{\mp 1}$. The exponent should not be left implicit in formulas whose normalization is already delicate.

2. In the sentence before equation (6.2), the definition of $\widetilde{L}^{\alpha-}$ contains L^α without a minus superscript. Please correct or explain this notation.

3. Sums such as that in equation (6.3) use q^α while $\alpha \in \mathbb{Z}/\ell\mathbb{Z}$. Choose the representatives $0, \ldots, \ell - 1$ explicitly, since q^α is not intrinsically defined on a residue class.

4. Equation (4.4) should write $z\mathbf{1}_N$ inside the determinant. The manuscript should also define precisely what normal ordering means for every matrix coefficient of this determinant and whether the determinant is ordinary, column, or quantum.

5. The notation R is used both for the N-dimensional dynamical particle-space objects in Section 3 and for the standard $\mathfrak{gl}_{\ell-1}$ matrix in equation (6.13). Distinct symbols would make the change of auxiliary spaces much easier to follow.

6. The statement that the Bethe subalgebra is maximal commutative should be accompanied by the hypotheses under which it holds and by a reference addressing the relevant truncated quotient. The word "central" should be reserved for elements commuting with the whole horizontal algebra.

7. The comparison with the quantum spin Ruijsenaars model of Uglov and of Lamers–Pasquier–Serban is presently qualitative. A concrete comparison of representation space, parameter dictionary, and the first Hamiltonian would substantially sharpen the novelty claim and clarify whether the two constructions are equivalent, dual, or genuinely different.

8. The line-interface interpretation at the end of Section 7 would benefit from the schematic referred to in the surrounding discussion; the compiled manuscript currently ends without such a figure. Please ensure that all intended material appears before the end of the document.

9. Bibliographic metadata should be checked. There are malformed arXiv and archive fields, duplicate year fields, and inconsistent punctuation and dash encodings. Several recent preprints should also be identified by stable publication data when available.

Recommendation

The proposed connection among necklace-quiver Coulomb branches, quantum-loop symmetry, and spin Ruijsenaars–Schneider dynamics is potentially novel and worth developing. The current manuscript nevertheless contains explicit contradictions in the Hamiltonian normalization and in the proposed Drinfeld–Jimbo realization, while mixed commutativity and physical-spin closure remain unproved. These points affect the principal results rather than peripheral details. I therefore recommend **major revision** and would ask to see a fully rederived version, with low-rank checks and a clear theorem/conjecture separation, before publication is considered.