Referee Report

Interior Zeros of Supersymmetric Indices

Recommendation: Major revision. The main complex-analytic observation is sound after a consistent choice of grading variable, but the present manuscript overstates what follows from finite series data and from a generic giant-graviton expansion. These issues affect several headline claims and should be resolved before publication.

Synopsis and overall assessment

The manuscript studies a one-variable specialization of a supersymmetric index

$$\mathcal{I}(q) = \sum_n d(n) q^{n/a}$$

through the coefficients of its plethystic logarithm. Introducing the integer grading variable $Q = q^{1/a}$, the authors write $\log \mathcal{I}(Q^a) = \sum_{m \geq 1} L_m Q^m$ and define arithmetic coefficients $\delta(m/a)$ by the Mobius transform in equation (7). The principal mathematical result is that an interior zero makes the logarithm singular at the nearest zero, so Cauchy–Hadamard fixes the exponential rate of L_m; the $d = 1$ term in equation (7) then dominates all $d \geq 2$ terms and gives the same exponential rate for $\delta(m/a)$. Equations (10) and (11) are the resulting zero-distance formula. Equations (12) and (13) supplement it with an argument-principle count and Jensen sum rule.

The manuscript applies this observation first to four-dimensional $\mathcal{N} = 1$ $SU(2)$ gauge theory with $2N_f$ doublets. The $N_f = 3$ s-confining case reproduces the plethystic logarithm of fifteen mesons, whereas finite series for $N_f = 4, 5$ display rapid growth and roots of the truncated index inside the unit disk. The second application concerns the $\mathcal{N} = 4$ $U(N)$ Schur index. Using the exact decomposition in equations (15) and (16), the authors associate the zeros of $S_N(Q)$ with finite-N giant-graviton corrections, propose the general scaling law (14), and derive the deformed-exponential scaling function (18). Numerical zeros are used to estimate the constant in equation (19) and motivate the refined large-N expression (20).

There is a worthwhile paper here. Relating the radius of convergence of the logarithm to the Euler-transform coefficients is simple but useful, and the attempt to turn it into a diagnostic for confined descriptions is physically interesting. The exact $U(N)$ Schur-index example also gives a suggestive meeting point between giant-graviton expansions and the classical zero theory of deformed exponentials. The Appendix A dominance estimate is essentially correct once all objects are defined as functions of Q, and the numerical values quoted for the displayed examples appear internally consistent.

At present, however, three different levels of statement are repeatedly merged: an exact theorem about an infinite coefficient sequence, numerical evidence from a finite truncation, and a heuristic interpretation of a generic giant-graviton expansion. In addition, the index is not generally a single-valued holomorphic function of q when $a > 1$, although zeros, winding numbers, and Jensen's formula are formulated in that variable. The claimed global minimum in equation (19), the error estimate in equation (20), and the conclusion that a one-giant sector alone produces the nearest zero are not established by the analysis supplied. The claims about infinite products are also mathematically too broad.

I therefore do not recommend acceptance in the present form. A revision that cleanly separates theorem, certified computation, and conjectural physics could be suitable for JHEP or a comparable high-energy theory journal. Given the amount of qualification and numerical documentation needed, a full-length article may serve the work better than the current Letter format.

Logical structure and principal assumptions

The argument depends on the following chain of assumptions and results.

- Equation (1) assumes a normalized index with vacuum term $\mathcal{I}(0) = 1$, a rational lattice of exponents, and the subexponential upper bound (2). These assumptions make $F(Q) := \mathcal{I}(Q^a) = \sum_n d(n)Q^n$ holomorphic for $|Q| < 1$.
- Equations (6) and (7) define the logarithmic and plethystic coefficients. If F has a zero of least modulus $\rho_Q < 1$, then the Taylor radius of $\log F$ is ρ_Q. Appendix A argues that Mobius inversion preserves the resulting rate, producing equations (10) and (11).
- Equations (12) and (13) assume that the same index is a single-valued holomorphic function on the contour used for the argument principle and Jensen's formula.
- The $SU(2)$ conclusions rely on exact finite series extracted from a matrix integral, together with root stability and exponential fits, to infer the infinite-order behavior required by equation (11).
- Section V assumes that a generic giant-graviton expansion has a one-brane coefficient $C \sim E$, that its higher-wrapping sectors do not remove the relevant zero, and that this is enough to infer equation (14). Section VI and Appendix C then test a much more specific realization of these assumptions using equation (16).

The first two bullets support a useful exact statement. The remaining steps need revision or additional hypotheses, as detailed below.

Major comments

1. **The zero problem and the argument principle must be formulated consistently in the Q plane.**

   For $a > 1$, equation (1) contains fractional powers of q. The single-valued holomorphic function supplied by the series is $F(Q) = \mathcal{I}(Q^a) = \sum_n d(n)Q^n$, not in general a holomorphic function on the q disk. Unless $F(Q)$ is invariant under $Q \mapsto e^{2\pi i/a}Q$—which the $N_f = 3, 5$ examples are not—it does not descend to a single-valued function of q.

   This causes a concrete problem in equations (9), (12), and (13). Distinct zeros in the Q plane can map to the same value under $q = Q^a$, their multiplicities need not be preserved by the proposed counting convention, and the contour integral $\frac{1}{2\pi i}\oint_{|q|=r} d\log\mathcal{I}$ is not defined without a branch and monodromy prescription. Saying merely that $q_* = Q_*^a$ does not repair the zero cardinality.

   The clean solution is to state the theorem, zero distance, argument principle, and Jensen formula entirely for $F(Q)$. One may subsequently quote the radial convention $\rho_q = \rho_Q^a$, but it should not be used to identify a set of q-plane zeros unless descent has been proved. Equations (4), (7), and (11) should also specify that limits are taken along the grading lattice and that powers are understood formally, or on one fixed branch.

2. **The "vanishing index" class contradicts the hypotheses of the theorem.**

   The third item in Section II says that the supersymmetric index may vanish everywhere and relates this to spontaneous supersymmetry breaking. Yet the paper uses $\mathcal{I}(0) = 1$ explicitly in Section III and Appendix A, and the definitions of $\log\mathcal{I}$, the plethystic logarithm (4), and the Mobius coefficients (7) all require a nonzero constant term. An identically zero function cannot belong to the same classification.

A superconformal index normalized by the vacuum and a Witten index in a theory with broken supersymmetry are different settings. The authors should remove the third class from this theorem or place it in a clearly separate discussion to which equations (4)–(13) do not apply. As written, the advertised trichotomy is logically inconsistent.

3. **Finite coefficient data do not establish the asymptotic criterion used for the $N_f = 4, 5$ claims.**

   Equation (11) is a statement about a limsup of an infinite sequence. Observing apparent exponential behavior through orders 60 or 150 does not prove that limsup is larger than one. Likewise, roots of a Taylor polynomial are not automatically roots of the underlying holomorphic function. Stability under a few truncation orders is useful numerical evidence, but it is not the rigorous obstruction asserted in Sections IV and VII. In particular, a finite prefix is compatible with a zero-free continuation having different later coefficients.

   This can be repaired with certified remainder control. The authors should display the specialized $SU(2)$ matrix integral, give the exact coefficient extraction algorithm, bound the tail on contours surrounding the quoted roots, and use Rouche's theorem or an interval-certified argument principle to prove the number of zeros. If such bounds are unavailable, the words "therefore," "rigorously," and the quoted fit accuracies should be replaced by a clear statement of numerical evidence.

   Reproducibility also needs improvement. The supplied Mathematica notebook begins the $N_f = 4, 5$ analysis from hard-coded coefficient lists; it does not derive them from the matrix integral or implement the stated Haar projection. Nor does it implement the Pade/argument-principle validation described in Appendix B. Comparing a growth fit with roots of the same truncated data is not an independent error estimate. The complete coefficient-generation and zero-certification calculation should be included.

4. **Equation (14) is not a consequence of a generic giant-graviton expansion as presently argued.**

   Section V reasons from a leading term $1 - CQ^E$ to a nearest zero at $C|Q|^E \sim 1$. At precisely that scale, however, an n-wrapping coefficient that grows as C^n contributes at order one after multiplication by Q^{nE}. Higher-wrapping sectors therefore cannot simply be called suppressed. They can move the zero, create several zeros, or remove it.

   Equation (18) illustrates the issue rather than resolving it. All values of n survive in the scaling function. At $\theta = 0$ it becomes e^{-w} and has no zero at all, despite having the same one-brane scaling variable. Thus a one-brane energy and coefficient do not by themselves imply the existence of an interior zero or equation (14).

   The authors should recast (14) as a conditional statement. Sufficient hypotheses would need to control the large-E scaling of every wrapping sector, produce a limiting function with an isolated zero, and establish uniform convergence near that zero. The exact $U(N)$ Schur-index family may satisfy such a proposition, but one example cannot confirm a general law for arbitrary giant-graviton expansions. The physically defensible interpretation is that the one-brane sector sets the scaling variable; the full multi-brane function produces the cancellation.

5. **The minimization in equation (19) and the remainder in equation (20) are not established.**

   There is a finite-N phase constraint that Appendix C does not discuss. Since $w_N = NQ^{N+1}$ and $\theta = \arg Q$, one has $\arg w_N = (N+1)\theta$ modulo 2π. The complex variable w_N and θ cannot be held independently fixed along an arbitrary sequence of ranks. It is plausible that an $O(1/N)$ adjustment of θ can match the phase near the minimizing branch, but that step and the required uniform convergence should be proved.

   The numerical procedure in the ancillary notebook also does not establish the global minimum written in (19). It follows one root returned from a single initial guess and minimizes that branch only over approximately $60^\circ \leq \theta \leq 78^\circ$. It neither enumerates all zero branches nor certifies that the tracked root is the nearest one for every θ. Similarly, the large-N table uses seeded Newton iterations and does not rule out a closer zero elsewhere. A global argument-principle scan with controlled truncation error, together with a phase-matching argument, is needed for the claim $w_\infty = 0.7378\ldots$.

Finally, the scaling limit leading to (18) establishes at most a leading limit. It does not derive the specific $\mathcal{O}((\log N)/N)$ remainder in (20). That estimate should be proved from finite-N coefficient and radial corrections, or weakened to $o(1)$. The description of w_∞ as the value at which the "one-giant sector first cancels the vacuum" should also be changed: at $|w| \simeq 0.738$ the one-giant term alone does not cancel the unit term; the higher terms in (18) are essential.

6. **Interior zeros do not obstruct infinite-product representations altogether.**

   The mathematical claim in the Introduction and Conclusions is false in its present form. Holomorphic and entire functions with zeros routinely admit Weierstrass or canonical product representations in which the zeros appear as explicit factors. An interior zero can obstruct a particular zero-free Euler product, or a finite plethystic exponential of standard free-multiplet single-letter indices, but it cannot rule out "an infinite product representation altogether."

   The authors should identify the precise product class relevant to the physical argument and prove that each allowed free matter factor is nonvanishing in the chosen disk under the stated charge and fugacity assumptions. The proposed implications for Rogers–Ramanujan, Askey–Wilson, and Macdonald identities should then be narrowed accordingly. This change is important both for mathematical correctness and for calibrating the claimed novelty.

7. **The physical diagnostic is a property of a chosen one-variable specialization, not of an index without qualification.**

   A four-dimensional superconformal index is naturally a multivariable function. Section IV sets $p = q$, apparently also unrefines all flavor fugacities, while Section VI uses the Schur specialization. Interior zeros can move, merge, or disappear when the fugacity slice is changed. The manuscript should therefore state precisely which specializations are being diagnosed and which reparametrizations preserve the assertion.

   The rational-grading assumption below equation (6) is also substantial: generic $\mathcal{N} = 1$ fixed points after a-maximization can have irrational charges. The present theorem does not directly cover them. Claims that the criterion rules out any free or s-confining infrared description should be restricted to the same unrefined slice and to the class of finite matter products proved to be zero-free there. A short discussion of accidental symmetries and decoupled free sectors would make the scope clearer.

8. **The novelty of the exact theorem needs sharper placement in the mathematical literature.**

   The proof of equations (10) and (11) combines the standard radius-of- convergence theorem for $\log F$ with the observation that the identity term dominates a Mobius or Euler transform whenever the growth rate is greater than one. This is elegant, but elementary. The manuscript should compare it with the literature on Euler transforms, Witt transforms, analytic combinatorics, and singularity analysis, rather than presenting it as an entirely new zero-detection mechanism on mathematical grounds. The potentially novel contribution appears to be its use for supersymmetric indices and the giant-graviton scaling example. Making that distinction explicit would strengthen the significance case for a high-energy theory journal.

Minor comments and presentation

1. Equation (2) should be written as an upper bound such as $|d(n)| \leq \exp(Cn^{\alpha})$, or with $\log(1 + |d(n)|)$. As printed, $\log |d(n)|$ is undefined when a coefficient vanishes; the $N_f = 5$ series itself has many zero coefficients.

2. The theorem would benefit from a boxed proposition listing all assumptions: $F(0) = 1$, rational grading, coefficient growth, the domain of holomorphy, and the lattice along which the limsup is taken. Appendix A should refer to the normalized holomorphic logarithm, rather than the "principal branch," which need not remain the elementary principal branch of the values of F.

3. The argument principle used after equation (12) requires that the contour contain no zero. The authors should state this and use a strict inequality in the counting function, or explain the limiting convention at a radius passing through a zero.

4. The explicit $SU(2)$ index integral, its Haar measure, the single-letter functions, and the flavor specialization should appear in the paper or End Matter. They are needed to reproduce Figure 1 and to check the grading statements for $N_f = 3, 4, 5$.

5. Table 1 labels a finite fitted value as a limsup. The first column should instead say "fitted growth rate" and give a fit window and an uncertainty. The same distinction should be maintained in the prose.

6. The assertions after Table 1 that the nearest zeros are simple and that the zero cardinality grows without bound as $r \to 1$ need either proofs or explicit numerical qualifiers. Positivity on the positive real axis does not by itself establish the rest of the zero distribution.

7. The phrase "as degeneracies must" in Appendix B is too quick. The $\delta(\nu)$ are Euler-transform exponents and need not be literal degeneracies of independent particles. If their integrality follows from the integral Euler transform of a unit-normalized integer series, that algebraic fact should be stated.

8. References 45 and 46 should be described carefully. The latter treats the real parameter range $0 < q < 1$, whereas equation (18) uses the complex unit circle. A direct reference for Sokal's full complex-parameter conjecture would be useful.

9. "A low order q-series expansion" in the Conclusions is misleading when some fits use order 1000 and there is no a priori stopping rule. "A finite high-order expansion, supplemented by error control" would be more accurate.

10. The introductory statement that a finite-system partition function necessarily has zeros should be restricted to a nonconstant polynomial. More importantly, the analogy with Lee–Yang zeros should not imply a phase transition: the manuscript's finite-rank interior zeros are analytic cancellations and are not shown to encode thermodynamic nonanalyticity.

Recommendation

The coefficient-growth lemma is useful and the applications are potentially interesting, but theorem-level status is currently assigned to claims that are branch-dependent, numerical, or heuristic. I recommend **major revision**. A publishable version should formulate zeros in the Q plane, remove the inconsistent vanishing-index class, certify or qualify the $SU(2)$ roots, make the generic giant-graviton law conditional, and establish the claims behind equations (19) and (20). It should also restrict the infinite-product and infrared statements to the fugacity specialization and product class actually proved.