Referee Report

Missing Descendants in the Carrollian Conformal Family

Recommendation: Not suitable for publication in its present form. The differential realization proposed in Section 2 is potentially useful, and a substantially reconstructed paper could merit reconsideration. At present, however, the manuscript does not establish that the added tower is a canonical or "complete" conformal family, that the formal generating series defines the Hilbert-space representations used later, or that the distributional Ward-identity solutions are exhaustive and well defined. The claims about irreducibility, null states, and massive flat-holographic sectors consequently go beyond what has been proved.

Synopsis and overall assessment

The manuscript starts from a simple and important observation. In the global Carrollian conformal algebra, $[P^0, K^0] = 0$. Hence the ordinary translation descendants $(P^0)^n\mathcal{O}_\mathrm{p}$ of a primary annihilated by K^0 are themselves annihilated by K^0; unlike in an ordinary conformal Verma module, K^0 does not lower that translation chain. The author then adjoins a new sequence $\mathcal{O}_n$ obeying $[K^0, \mathcal{O}_n] = \mathcal{O}_{n-1}$ in equation (2.6), includes the translation descendants of every level in equation (2.8), and packages the boost and K^0 chains into generating operators with continuous variables β (or β^a) and κ.

This construction leads to explicit first-order differential actions of the global algebra on the enlarged spaces $(u, z; \beta, \kappa)$ in equation (2.24) and $(u, z^a; \beta^a, \kappa)$ in equation (2.63). The manuscript then introduces flat formal pairings, derives principal lines and conjugate labels, exponentiates the vector fields, classifies the signs of κ and the $\kappa = 0$ orbits, and evaluates quadratic and higher Casimirs. In Fourier space the quadratic Casimir becomes $\rho\kappa - \beta^2$ in two dimensions and $\rho\kappa - \vec{\beta}^2$ in three dimensions. The latter is identified with m^2 as motivation for a possible description of massive states.

Section 3 applies the differential actions to global two-point Ward identities. It separates the cases in which both κ_i are nonzero, both vanish, or only one vanishes. The proposed solutions include shifted non-contact supports, point-supported electric branches, and, in three dimensions, additional distributions on real blow-ups that retain angular data at a collision. A notable feature is the appearance of arbitrary functions of invariants even at two points.

The topic is timely and the central algebraic idea is worth investigating. The explicit actions (2.24) and (2.63) organize a large set of transformations compactly, connect naturally to continuous boost-charge bases, and suggest a useful way to expose nonzero quadratic-Casimir sectors. The separation of generic, zero-κ, and mixed Ward problems is also sensible. In particular, writing the B^a and K^0 constraints as shifted support conditions makes the geometric content of the generic solutions visible.

The main difficulty is that several logically distinct constructions are treated as if they were automatically equivalent. An algebraic extension of a highest-weight module, a formal power-series generating function, a direct-integral representation with real continuous labels, a local operator family, and a Hilbert-space representation all require different completion and domain data. Those data are not supplied. In Section 3 the formulas are too distributionally singular for the omitted derivations to be optional: generic coordinate charts miss degenerate strata, products and restrictions of Dirac distributions require a test-function prescription, and the blow-up branches live on spaces with unspecified measures and lifted group actions. These issues affect the principal conclusions rather than only the presentation.

Logical structure and principal assumptions

The argument relies on the following chain of assumptions.

- Equation (2.6) is taken to supply the descendant direction that a Carrollian conformal family must contain, and equations (2.6)–(2.8) are taken to define a complete family rather than one possible extension of the translation-generated module.
- The ordinary generating series, such as equation (2.14), is treated both as a formal Taylor series whose coefficients recover discrete states and as a continuous eigenbasis for arbitrary real β and κ.
- The flat pairings (2.26) and (2.64), together with separately chosen pairings on $\kappa = 0$, are assumed to provide the adjoint operation and the relevant principal series without an explicit function space, invariant measure, or operator-domain analysis.
- A nonconstant differential action of a higher Casimir is taken to prove reducibility, while constancy of the displayed Casimirs plus an asserted stabilizer result is taken to prove irreducibility at $\kappa = 0$.
- Solutions found in generic invariant charts, supplemented by a point-contact ansatz, are taken to exhaust the distributional solutions of all global Ward identities.
- Radial delta functions and angular data on real blow-ups are taken to define additional covariant correlators without specifying the density, line bundle, or lifted action on the blown-up spaces.

Each assumption could be made into a precise and interesting construction, but none follows merely from closure of the displayed first-order differential operators. The manuscript needs to distinguish what is defined, what is proved, and what is conjectured.

Major comments

1. **Equation (2.6) defines an extension; it does not establish a missing or canonical part of a conformal family.**

   A highest-weight conformal module generated by a primary is, by definition, generated by the chosen raising subalgebra, here including translations. Equations (2.4) and (2.5) show that K^0 acts trivially on the temporal translation chain. They do not imply that K^0 must be surjective onto that chain, nor that vectors with prescribed K^0 preimages must be adjoined. The operators $\mathcal{O}_n$ in equation (2.6) are not descendants of $\mathcal{O}_\mathrm{p}$ under the original descendant-generating algebra; they are new vectors defining an extension in the opposite direction.

   There can be no completeness claim until the relevant category and universal property are stated. One may choose no such extension, the one-dimensional chain displayed here, chains with multiplicities, nontrivial homogeneous pieces in the K^a actions, quotients, or further indecomposable extensions. Closure of (2.24) and (2.63) verifies that one economical differential realization has been written down; it does not classify all modules containing the original primary.

   The author should formulate the construction as an explicit module: give its generators and relations, weight-space multiplicities, submodules and quotients, and a character. If it is induced, coinduced, injective, or otherwise universal in a specified category, that statement should be proved. If no such property is intended, "extended Carrollian module" would be accurate, while "missing descendants" and "complete family" should be presented as motivation rather than theorem. The comparison with the K^0-lowered sequence in the recent work cited near the end of Section 1 should be made at this module-theoretic level, not only through the value of the quadratic Casimir.

2. **The formal generating series and the continuous Hilbert-space realization have not been connected.**

   Equation (2.14), and its three-dimensional counterpart (2.56), is an ordinary power series in nonnegative powers of the representation variables. Equation (2.15) recovers discrete vectors by derivatives at the origin. Later, the same variables range over the real line or plane, are integrated with Lebesgue measure in (2.26) and (2.64), and serve as continuous spectral labels. These uses are not automatically compatible. No convergence region or topology for the series is given; monomials are not in the proposed L^2 spaces; and evaluation of arbitrary derivatives at the origin is not a continuous functional on L^2. A rigged space can accommodate generalized eigenvectors, but it must be constructed.

   There is a related measure-theoretic problem at $\kappa = 0$. In $L^2(\mathbb{R}, d\kappa)$, functions supported on the single point $\kappa = 0$ represent the zero vector. Thus (2.34) and (2.71) do not simply restrict (2.26) and (2.64) to an invariant Hilbert subspace; they introduce separate singular-measure representations. It is permissible to form a direct sum or direct integral containing such orbit measures, but this must be stated, and the resulting adjoints and dual dimensions must be derived for each summand.

   The author should specify a dense invariant test space, its dual, the measure on every orbit, and domains for all unbounded differential generators. Anti-Hermiticity should then be checked for the full set of generators, including boundary terms at singular orbit strata. Until this is done, the principal lines (2.29), (2.35), (2.67), and (2.72) are properties of selected formal pairings, not intrinsic consequences of the algebra. The same caveat applies to the conjugation formulas and to the later interpretation of vanishing kernels as zero-norm states.

3. **The Casimir and irreducibility conclusions require an actual representation-theoretic proof.**

   The quadratic-Casimir calculations (2.44) and (2.80) are transparent and useful. The subsequent conclusions are much less secure. The long operator in (2.84) is presented without first giving the quartic Casimir as an element of the enveloping algebra, fixing its ordering convention, or deriving the displayed differential expression. The statement that it commutes with every generator should be demonstrated, preferably in a reproducible appendix or ancillary calculation. The same applies to the use of the Pauli–Lubanski operator (2.47)–(2.48).

   More importantly, a differential central operator that is not visibly a scalar can motivate a spectral decomposition, but reducibility depends on its domain and spectral theory. Conversely, constant values of the two displayed Casimirs do not prove irreducibility. The manuscript repeatedly invokes a "stabilizer analysis," yet it never displays the stabilizer, its representation, the induction procedure, or the hypotheses under which the induced representation is irreducible. Casimirs can fail to separate irreducible representations, especially for non-semisimple groups.

   The author should construct the relevant induced representations orbit by orbit, identify the little groups and their unitary representations, and state the measure and global group being represented. Only then can the reducibility and irreducibility assertions in the paragraphs following (2.48) and (2.85) be retained.

   The massive interpretation in (2.86) must also be separated from the algebraic calculation. Relabeling a quadratic Casimir as m^2 and solving $\rho = (m^2 + \vec{\beta}^{\prime 2})/\kappa$ for real variables does not construct a positive-energy unitary massive Poincare representation or a boundary holographic dictionary. The normalization and sign of the Poincare Casimir, the translation generators, orbit, little group, and physical inner product all need to be identified explicitly. A work in progress cannot carry this central inference. In the present paper these sectors should be described only as candidates with positive quadratic Casimir.

4. **The Ward-identity classification is not demonstrably exhaustive.**

For formulas as intricate as (3.10), (3.26), (3.36), and (3.45), the full Ward equations and their solution cannot be replaced by a statement that they follow from (2.24) or (2.63). The reader needs to see the characteristic equations, distributional constraints, Jacobians, and the action of every remaining generator.

The generic formulas also divide by quantities that can vanish inside the stated orbit configuration. In two dimensions, (3.8)–(3.10) divide by $\beta_1 - \beta_2$ and $\kappa_1 + \kappa_2$, although the covariant support can be written without either division as (3.11). The locus $\beta_1 = \beta_2$, $\kappa_1 + \kappa_2 = 0$ is therefore a singular chart of the same non-contact constraint, not automatically an empty branch. In three dimensions, (3.23)–(3.26) divide by both $\kappa_1 + \kappa_2$ and $\vec{\beta}_1^{\,2} - \vec{\beta}_2^{\,2}$. When both vanish, the undivided B^a and K^0 constraints lose rank and must be solved anew. Similar singular loci occur in the mixed invariants (3.19) and (3.43), including the chart boundary at vanishing second boost label.

It is not enough to discuss only the conjugate point inside these degeneracies. A distributional PDE can acquire solutions supported on lower-rank strata, as well as derivatives of delta functions. The three coarse cases "both nonzero," "both zero," and "mixed" do not by themselves classify these strata. The author should stratify parameter space by the ranks of the Ward vector fields and constraint maps, solve every stratum, and prove that no additional distributions are allowed. Each candidate invariant should have a stated domain and an independence check. Direct substitution of every final formula into all Ward operators, with distribution identities shown, would provide an essential verification. Until then, the words "all," "only," and "complete" in the correlator classification are not justified.

5. **The kernels in representation labels and their specialization to conjugate operators are not defined consistently as distributions.**

   The manuscript alternates between treating β, κ, Δ, and l as fixed external labels and treating the two-point function as a delta-normalized kernel in those labels. A precise test-function space and measure are needed. In particular, Δ lies on a continuous principal line, so the symbols $\delta_{\Delta_1,\Delta_2}$ in (3.15), (3.36), and (3.42) cannot be Kronecker deltas in a normalized continuous basis. One should state an equality selection rule separately or use the appropriate Dirac distribution in the spectral parameters.

   Several displayed "conjugate-pair" formulas are not literal restrictions of the preceding kernels. For example, the support $\delta(\beta_1)\delta(\beta_2)$ in (3.16) would yield a square of a delta distribution after imposing $\beta_2 = -\beta_1$, whereas (3.17) retains one delta. Likewise, the two independent vector deltas in (3.37) do not pull back to the single delta written in (3.40), and the three radial deltas in (3.39) cannot simply be reduced to the support written in (3.41). These pullbacks are non-transverse and are generally undefined. If the intention is to factor out an overall delta-normalization volume after smearing, that convention and its Jacobian must be shown explicitly.

   Nonlinear supports require equal care. The distribution $\delta(\vec{\beta}_1^{\,2} - \vec{\beta}_2^{\,2})$ in (3.35) has a critical point at the zero labels, while angular ratios in (3.36) are undefined there. A separate extension across that locus is not unique without a scaling or regularization prescription. Consequently, the statements that generic nonzero boost-label conjugate states are null and that only particular zero modes have nonzero norm do not yet follow. Such statements additionally require a nondegenerate physical two-point pairing and a positivity or quantization framework, which the manuscript correctly acknowledges has not been constructed.

6. **The real-blow-up branches need a mathematical definition and a physical interpretation before they can count as correlators.**

   Replacing the origin of $\mathbb{R}^2$ by an exceptional circle can be a useful resolution of directional data, but equations (3.29)–(3.41) do not yet define distributions on that resolution. A symbol such as $\delta(r_z)$ depends on the chosen density: it has one meaning against $dr_z\, d\phi_z$ and a different, singular meaning relative to the pullback of $d^2z = r_z dr_z\, d\phi_z$. This choice is also what determines the claimed scaling powers of u. The manuscript specifies neither the density nor a pushforward to the original plane.

The complete generator action must be lifted to the blown-up manifold and shown to preserve the relevant boundary and test-function space. Checking rotation weights of an angular factor is not sufficient: normal vector fields can create derivatives of radial deltas, and finite conformal maps can act nontrivially on the exceptional directions. For the simultaneous blow-ups in (3.38), the author should define the resulting manifold with corners, the three boundary densities, the Spin(2) line bundle carrying the fractional angular powers, and the action on each boundary face. The asserted invariance of the two angle differences in (3.39) should then be derived from this lifted action.

Finally, these branches are expressly not distributions on the original configuration and representation spaces. The paper must explain what observable, defect, directional limit, or enlarged operator algebra makes them physical two-point functions. Without such an interpretation they are mathematically auxiliary boundary distributions, not additional local CarrCFT correlators on the original spacetime. Claims about state pairings and the completeness of physical correlators should be restricted to the ordinary configuration space unless this enlargement is justified.

Further comments and requested clarifications

1. The paper should distinguish consistently among a primary module, an extended module, a formal generating object, and a local field over an enlarged parameter space. At fixed nonzero κ, the generating operator is not a D eigenoperator because of the $\kappa\partial_\kappa$ term in (2.21); calling Δ its scaling dimension without qualification is potentially misleading.

2. The claim that the algebra fixes the discrete K^a actions in (2.17)–(2.19) and (2.55) assumes the displayed one-dimensional weight spaces and excludes homogeneous intertwiners or internal multiplicities. These assumptions should be stated before the word “fixes” is used.

3. Equation (2.74) should read “greater than or equal to zero.” The denominator vanishes when $\vec{k}$ and $\vec{z}$ are parallel and $\vec{k} \cdot \vec{z} = 1$. The text may then restrict to the regular chart on which it is strictly positive.

4. The finite transformations in Tables 1 and 2, the closure of (2.24) and (2.63), and the centrality of (2.84) should be accompanied by enough intermediate algebra to permit verification. A short symbolic notebook would be valuable, but the manuscript must still state conventions and key identities explicitly.

5. Absolute values raised to complex powers, as in (3.10), require a branch and a distributional prescription at zero. The manuscript gives a convention for complex angular factors in three dimensions but not for all radial complex powers or their analytic continuation.

6. The normalization of shifted delta functions should be uniform. For every formula, it would help to display first the undivided covariant constraint and then the Jacobian used to solve it in a particular chart. This would also make clear why apparently similar factors differ between (3.10), (3.21), (3.26), and (3.45).

7. Section 3 should state whether derivatives of contact deltas are excluded by an assumption, by scaling, or by the Ward equations. Starting from a zeroth-order point delta is an ansatz, not an exhaustive classification of point-supported distributions.

8. The relation to the literature should be sharpened. The manuscript cites established boost-charge representations, higher-dimensional Carrollian modules, recent K^0-lowered sequences, and induced

representations. A table comparing generators, module category, Casimirs, measures, and submodule structure would make the actual novelty much clearer. Results delegated to “work in progress” should not be used to support the present paper’s principal physical claims.

9. The abstract should say that the manuscript *proposes* or *constructs* an extended representation and candidate massive sectors. Statements that it constructs “the complete” representation, determines all orbit properties, or identifies physical massive states should await the missing proofs. The tense also shifts from “We construct” to “We derived” in the abstract and should be made consistent.

Novelty, significance, clarity, and suitability

The algebraic observation behind the paper is simple but potentially fruitful, and the explicit differential realizations may be useful to the Carrollian and flat-holography communities. If placed on a precise module- and representation-theoretic footing, the extra κ direction could provide a helpful bridge between familiar zero-Casimir boundary modules and more general induced representations. The attempt to classify two-point kernels across different orbit types is also more ambitious than a routine extension of known magnetic and electric correlators.

At present, however, the novelty is difficult to assess because the central object is not characterized relative to known extensions, induced modules, or the recent K^0-lowered construction cited by the author. The physical significance is likewise prospective: no field theory realizes the proposed nonzero-κ sectors, no positive physical inner product is available, and no massive holographic dictionary is constructed. The manuscript is generally readable at the motivational level, but the most important derivations are replaced by assertions just where singular distributions and degenerate group orbits demand additional detail.

For a high-energy theory journal such as JHEP or PRD, the correctness issues above must be resolved before publication. A viable new submission would define the extended module and its function spaces precisely, construct the orbit representations and measures, provide a verifiable Casimir analysis, solve the Ward identities stratum by stratum, and either give a rigorous blow-up framework or omit those branches from the physical classification. The claims about massive particles should remain explicitly conjectural unless an actual Poincare and holographic dictionary is included.

Publication recommendation

I therefore recommend rejection in the present form, with encouragement to resubmit a substantially reconstructed and more narrowly stated paper. The differential representation is a promising starting point, but the main claims presently depend on unproved completion, measure, irreducibility, and distributional-exhaustiveness assumptions. Resolving those points would materially change both the mathematical content and the physical conclusions of the work.