Referee Report

Covariant variation and its applications

Recommendation: Major revision. The tensor-algebraic core is useful and, in its present formulation, largely coherent. However, the principal operator is an instance of the known metric or reductive Lie derivative, while the manuscript does not yet isolate sharply what is new. More importantly, the proposed applications to Maxwell and p-form gauge potentials are not shown to descend to a reduced gauge phase space. The horizon and general-dimensional "helicity fluxes" therefore remain formal Hamiltonians for assumed boundary variables, rather than established observables of the stated bulk theories. These issues affect the main physical conclusions and should be resolved before the paper is suitable for JHEP or PRD.

Synopsis and overall assessment

The manuscript studies the first-order tensor derivation

$$\Delta_X = \mathcal{L}_X + \rho(S_X) = \nabla_X + \rho(A_X), \qquad S_X = \tfrac{1}{2} g^{-1}\mathcal{L}_X g, \qquad (A_X)_{\mu\nu} = \nabla_{[\mu} X_{\nu]}.$$

On ordinary tensors, the symmetric part of ∇X compensates the metric variation under the Lie derivative. As a result, Δ_X is a derivation, commutes with contractions, annihilates the metric and volume form, and commutes with the Hodge star. Equations (2.43a)–(2.43b) give its central algebraic feature,

$$[\Delta_X, \Delta_Y] - \Delta_{[X,Y]} = \rho\big(-[S_X, S_Y]\big).$$

The manuscript interprets this non-closure as a local orthogonal rotation and calls it an anomaly.

Sections 3 and 4 develop the tensor-algebra properties, the lack of C^∞-linearity in X, commutators with other differential operators, the action on forms, the relation to the Kosmann derivative, and the metric-frame lift. This part also embeds ∇_X, $\mathcal{L}_X$, and Δ_X in two- and three-parameter families. The two-parameter curvature formula (4.28) correctly reproduces the Riemann curvature, the covariant-variation non-closure, and the flat Lie-derivative point at the advertised special values. The geometric explanation in Section 4.3 is particularly helpful: ordinary tensors, Lorentz tensors or spinors, and mixed fields carry different representations of the same metric-dependent frame lift.

Section 5 turns to physical applications. For radiative Maxwell data at null infinity, the stress-tensor superrotation flux acts by a metric-compensated boundary transformation, and the commutator of two such transformations is a two-dimensional polarization rotation. The paper then proposes an analogous construction at a finite Gaussian-null horizon, extends the kinematics to an arbitrary covariant tensor in d dimensions, and specializes to a free p-form in $d = 2p + 2$. With an assumed radiative symplectic form, the orthogonal rotation is assigned a quadratic Hamiltonian; in a nonzero-frequency mode expansion this Hamiltonian becomes a weighted difference of occupation numbers for conjugate polarizations.

There is much to like in the organization of the algebra. The equivalence between the two forms of Δ_X, the commutator calculation, the half-density explanation of the extra divergence term at null infinity, and the distinction between a single four-dimensional duality rotation and a general $\mathfrak{so}(d-2)$ polarization rotation are all valuable. The paper also makes several important qualifications: Δ_X is not an affine connection in its first argument; an arbitrary tensor rank does not specify dynamics; the general-dimensional phase space is an additional input; and the mode argument excludes the soft sector.

The remaining problems are nevertheless structural. The paper itself shows in equations (3.19)–(3.21) that Δ_X does not generally commute with the exterior derivative. This directly raises the question whether its action on a gauge potential preserves gauge orbits. That question is not answered before the operator is used as a Hamiltonian transformation of Maxwell and p-form potentials. In addition, several boundary symplectic forms are asserted without a complete constraint and gauge reduction, and an operator non-closure on fields is sometimes treated as if it already fixed a Poisson or quantum flux algebra. It does not: boundary, soft, central, and domain contributions must still be determined.

Logical structure and principal assumptions

The argument rests on the following chain.

- A metric-dependent lift of a vector field to the orthonormal-frame bundle defines Δ_X on ordinary tensors and the Kosmann derivative on Lorentz tensors or spinors. The curvature of this lift is represented by $-[S_X, S_Y]$.
- After a null-boundary limit and a choice of conformal or density weight, the induced transformation is assumed to act on the radiative potential within a fixed reduced phase space.
- The Maxwell symplectic form (5.12), the model tensor symplectic form (5.48), or the p-form symplectic form (5.60) is assumed to be non-degenerate after all constraints, residual gauge transformations, soft modes, and endpoint data have been treated.
- A quadratic functional is then inferred from its action on the displayed radiative variables. The commutator curvature is identified with a second Hamiltonian that rotates the transverse polarization.
- The resulting local $\mathfrak{so}(d-2)$ generator is interpreted as a helicity flux and, in the p-form mode basis, as a difference of conjugate-polarization occupation numbers.

The first step is mathematically well motivated. The later steps require additional dynamical and global input. In particular, a transformation of a gauge-fixed representative is not automatically a vector field on the quotient phase space, and equality of two Hamiltonian vector fields determines their Hamiltonians only up to phase-space constants and boundary charges.

Major comments

1. **The known metric-lift construction must be separated much more sharply from the new claims.**

   The paper identifies Δ_X with the metric Lie derivative and relates it to the reductive and Kosmann lifts. It also states that the curvature formula in equation (4.7) is known in the recent metric-lift literature. Consequently, metric preservation, the Hodge-star property, and the basic non-closure in equations (2.43a)–(2.43b) cannot by themselves carry the novelty of a high-energy theory paper. The two-parameter family in Section 4.4 is a neat packaging, but most of its special points follow directly from linear combinations of ∇_X, $\mathcal{L}_X$, and Δ_X.

   The authors should provide a precise accounting of what is already known for the metric/reductive Lie derivative, what was established in their earlier null-infinity papers, and what is first proved here. A compact proposition-by-proposition comparison would be more useful than the present broad literature list. The genuinely new candidates appear to be the finite-horizon construction, the arbitrary-dimensional boundary reduction, and the p-form mode interpretation; those are exactly the parts that need the most additional proof.

   The word “anomaly” also needs discipline. The quantity $-[S_X, S_Y]$ is the classical curvature of a non-flat lift, not a quantum anomaly. Citing the Adler–Bell–Jackiw anomaly and noting a relation to helicity does not establish an equality of cocycles, effective actions, or anomalous Ward identities. I recommend calling it the "non-closure curvature" throughout the mathematical sections. If "anomaly" is retained in the applications, the paper should state precisely in what cohomological or quantum sense the term is intended and derive the claimed relation rather than suggest it by terminology.

2. **The proposed transformation is not shown to descend to gauge equivalence classes.**

   This is the most important correctness issue. Equations (3.19)–(3.21) explicitly show that

   $$[\Delta_X, \mathrm{d}] = [\rho(S_X), \mathrm{d}] \neq 0$$

   for a generic vector field. Therefore, under the gauge transformation $B \mapsto B + \mathrm{d}\Lambda$,

   $$\Delta_X(B + \mathrm{d}\Lambda) - \Delta_X B = \mathrm{d}(\Delta_X \Lambda) + [\Delta_X, \mathrm{d}]\Lambda$$

   is not generally exact. Thus the slotwise tensor transformation of a gauge potential need not map a gauge orbit to a gauge orbit. Metric compatibility and preservation of the Hodge star do not cure this problem.

   Section 5 nevertheless applies Δ_X to a Maxwell potential and to a p-form potential, culminating in equations (5.43), (5.46), and (5.57)–(5.64). The authors must show, for each application, that the boundary transformation is tangent to the constraint surface and descends to the quotient by residual gauge transformations. This requires an explicit treatment of radial gauge, the leading Gauss constraint, the u-independent residual gauge parameters, and the soft and zero-frequency sectors. Alternatively, the analysis may be restricted from the outset to a completely gauge-fixed, nonzero-frequency polarization space, with a proof that the restriction is preserved. Without one of these routes, the quadratic functionals can depend on the chosen potential representative and cannot yet be called physical fluxes.

   This point is especially visible for the local rotation $\delta_h B_{A_1 \ldots A_p} = -p h_{[A_1}{}^{C} B_{|C|A_2 \ldots A_p]}$. For angle-dependent h, multiplication by h does not generally send an exact angular form to an exact angular form. The authors should compute its action on residual gauge directions and show that the proposed Hamiltonian (5.62) is gauge invariant on the stated phase space.

3. **The finite-horizon phase space and flux algebra need a full derivation.**

   Equations (5.12)–(5.19) pass quickly from the pullback of the covariant presymplectic current to a fundamental commutator and then to the action of stress-tensor fluxes. Several ingredients are suppressed: the topology and boundary of the cross-section N, the residual Maxwell gauge group, the kernel of the null symplectic form, the role of the integration function $\varphi(\Omega)$ in (5.14), the endpoint conditions at $u = \pm\infty$, and possible corner terms. These are not cosmetic because both the symplectic inverse in (5.13) and the functionals in (5.17) contain precisely the zero-mode information on which such choices act.

   The restriction to $\nabla_A \mathcal{Y}^A = 0$ is sensible and is preserved by the Lie bracket. On that subalgebra, equation (5.30) is formally skew-adjoint and agrees with the displayed action of the flux. However, showing that $\mathcal{F}_{\mathcal{Y}}$ acts on $\mathcal{A}_A$ does not by itself prove

   $$\{\mathcal{F}_{\mathcal{Y}}, \mathcal{F}_{\mathcal{Z}}\} = \mathcal{F}_{[\mathcal{Y},\mathcal{Z}]} + \mathcal{O}_{o_q(\mathcal{Y},\mathcal{Z})}.$$

   The two sides can differ by a phase-space constant, a soft charge, or a boundary term. The paper presently calls (5.33) a "possible" extension, which is appropriately cautious, but the abstract and conclusion speak as though the horizon helicity flux has been derived. The Poisson bracket of the regulated Hamiltonians should be computed directly, including all endpoint and corner contributions, or the stronger claims should be withdrawn.

There is a related fixed-background issue. A divergence-free $\mathcal{Y}$ need not be Killing, so it changes q_{AB} even while preserving its area form. The manuscript should explain whether the relevant phase space includes q_{AB} and its conjugate data, or whether the metric compensation is a field-space connection on a family of fixed backgrounds. These are different Hamiltonian setups.

4. **The arbitrary-tensor "helicity flux" is conditional on a postulated phase space and should be presented as such.**

   The kinematical reduction in equations (5.38)–(5.46) is useful. In particular, the manuscript correctly notes that the angular component is closed only after the radial projection condition (5.42), and it cleanly separates the conformal weight from the antisymmetric spin action in (5.43). These statements do not, however, determine the dynamics or the reduced symplectic structure of a general rank-k tensor.

   Equation (5.48) is introduced as the "simplest choice." It is not a universal consequence of a bulk action. Trace constraints, Young projections, compensators, gauge constraints, and normalization change the physical inner product and sometimes the set of radiative variables. The condition $\Delta = m/2$ in (5.49) and the Hamiltonians (5.50)–(5.52) are therefore statements about the postulated model phase space, not about an arbitrary spinning field.

   The authors should either select a definite dynamical theory and derive its radiative phase space, or recast this subsection as a conditional proposition: given a non-degenerate symplectic form of the form (5.48), a specified invariant inner product, the projection condition (5.42), and stated endpoint conditions, the displayed transformations are Hamiltonian. The abstract and conclusion should then use the same conditional language. In higher dimensions, h_{AB} generates a local $\mathfrak{so}(d-2)$ polarization rotation; calling every such local generator "helicity" obscures the important distinction that the paper itself later makes.

5. **The p-form symplectic and mode analysis needs the missing constraint reduction.**

   The free action (5.57) and presymplectic current (5.59) are a good starting point, but equation (5.60) is asserted after "solving the leading constraints" without showing that calculation. This is the crucial step. The revised paper should give the full large-r expansion in radial gauge, identify all leading constraints, determine the residual gauge group, display the quotient or gauge fixing that makes (5.60) non-degenerate, and derive its sign and factor $1/p!$ from the chosen null-boundary orientation. Endpoint and soft contributions should be retained until the assumptions that remove them are stated.

   Only after this reduction can equations (5.62)–(5.64) establish a physical Hamiltonian. The mode expansion (5.66), oscillator algebra (5.67), and number-operator formula (5.69) then require a globally specified polarization bundle and inner product. The manuscript acknowledges that an angle-dependent diagonalizing frame may exist only locally and that the zero-frequency sector is excluded; these qualifications should be moved into the statement of the result, not left after it. The occupation-number conclusion is valid, at most, on the gauge-fixed nonzero-frequency Fock sector under the stated normal-ordering choice.

   Finally, for $p > 1$ a general h_{AB} is not a single duality or helicity generator. It is an element of $\mathfrak{so}(2p)$ with several Cartan weights, and an angle-dependent h is not a global internal symmetry of the bulk action without further structure. The careful distinction made after equation (5.69) should govern the terminology in the abstract, Section 5 heading, and conclusion.

6. **The extension to connections needs a single geometric convention, and equation (4.40) contains a concrete inconsistency.**

   The manuscript is appropriately explicit that an affine connection is not an associated tensor and that the metric lift does not uniquely prescribe a fixed-coordinate coefficient variation. Appendix B then lists several possible definitions and a larger family. This breadth presently makes it difficult to tell which object is used in the main results. I suggest selecting one geometric object – most naturally the operator variation $\Delta_X \nabla = \mathcal{L}_X \nabla - \nabla S_X$ – and distinguishing it from all optional prolongation conventions. The remaining family can be recorded as a comparison, but its members should not all be called the covariant variation of the Levi-Civita connection.

There is also an unambiguous sign error in the three-parameter discussion. The text preceding equation (4.40) chooses $(\alpha, \beta, \gamma) = (0, -1, 1)$, and the right-hand side of (4.40) is $\nabla_{\mathcal{Y}} + \rho(-\phi_{\mathcal{Y}} + A_{\mathcal{Y}})$. By definition (4.38a), this is $D_{\mathcal{Y}}^{(0,-1,1)}$, whereas the left-hand side of (4.40) is printed as $D_{\mathcal{Y}}^{(0,1,1)}$. The latter would contain $+\phi_{\mathcal{Y}}$. This should be corrected, and all special points of the three-parameter family should be checked against (4.38a) and the curvature formula (4.46).

Minor comments

1. The non-C^∞-linearity displayed in equations (3.3)–(3.5) is conceptually essential. It should be stated next to the initial definition, before language suggestive of transport or a connection is introduced.

2. Please distinguish consistently between S_X and A_X as endomorphisms and as lowered two-tensors. Several formulas rely on signs from the covector representation ρ, and an explicit convention near equation (2.35) would prevent ambiguity.

3. Equation (5.12) should be written as a bilinear form $\Omega(\delta_1 \mathcal{A}, \delta_2 \mathcal{A})$ rather than with the same variation in both slots. Equation (5.13) should state whether it is a Poisson bracket or a quantized commutator and how the null zero mode was removed when inverting ∂_u.

4. State explicitly that the horizon cross-section N is compact and without boundary, if that is assumed. Otherwise the integrations by parts used for skew-adjointness and the flux algebra acquire boundary terms.

5. The role of $\varphi(\Omega)$ in equation (5.14) should not be dismissed by setting it to zero without explaining whether it is gauge, constrained data, or a physical soft mode.

6. Reserve "helicity" for the constant four-dimensional duality charge or for specified Cartan generators of the massless little group. "Local transverse polarization rotation" is more accurate for a general $h_{AB}(\Omega)$.

7. The equality between the non-closure action on fields and a flux commutator should always be stated modulo central and boundary terms until the Hamiltonian bracket has been computed. Normal-ordering terms should be included when quantum commutators are used.

8. Large blocks hidden with `\iffalse`, duplicate draft equations, and obsolete labels should be removed from the submitted source. They make the already long appendices harder to verify and increase the risk of stale cross-references.

9. The prose would benefit from a careful edit. Recurring expressions such as "we can expect the emergence," "the story is not as perfect," and "it is easy to find" should be replaced by precise statements of hypotheses and conclusions. The abstract should distinguish proved identities, conditional phase-space constructions, and conjectures.