Referee Report

Soft charges and zero modes at null boundaries

Recommendation: Reject in present form, with reconsideration only after a fundamental revision. The paper proposes a potentially useful way to organize residual modes of null-surface constraint operators and their Regge–Teitelboim improvements. The present argument, however, does not establish the claimed Hamiltonian symmetry or charge algebra. The central construction treats an operator with an admitted kernel as second class and invertible, omits the solvability condition for the multiplier equation, and does not specify the operator domain or boundary conditions that determine the kernel. In the reduced-phase-space calculation there is a definite sign error in equation (36), the Hamiltonian vector field is incomplete, and boundary contributions are discarded even though they are precisely where a central extension can reside. The Maxwell–Pontryagin entry in the paper itself supplies a direct test that the asserted vanishing requires more analysis. Finally, gauge invariance, finiteness, conservation, and the claimed extension to finite black-hole horizons are not demonstrated. These issues concern the principal result, not presentation alone, and preclude publication in PRD or JHEP in the current form.

Synopsis and overall assessment

The paper studies canonical evolution on a null hypersurface Ξ. Its starting point is a set of primary constraints χ_α whose distribution-valued Poisson-bracket kernel is denoted by $\Omega_{\alpha\beta}$ in equation (1). Equations (2)–(5) postulate a residual kernel parametrized by arbitrary functions on the codimension-one boundary $\partial\Xi$. The corresponding map $P^\alpha{}_I$ fixes the radial dependence of a zero mode while leaving its angular profile free. Because the multiplier equation (8) then has a homogeneous part, equation (10) associates the arbitrary boundary functions with residual transformations rather than ordinary bulk gauge transformations.

Equations (11)–(14) smear the constraints with those modes, improve the resulting functional by a boundary charge, and allow in principle for a central extension. Section 3 specializes the discussion to scalar, Maxwell, Maxwell–Pontryagin, and Yang–Mills systems in Bondi coordinates. For the local first-order operators in equations (16)–(22), the zero-mode map is represented by $U^\alpha{}_I$. The transformation in equation (24) is a shift of the affected boundary field. Tables 1 and 2 collect the constraint data, radial zero modes, and coefficients entering the charge. After imposing fixed leading boundary coefficients in equations (32)–(33), the paper integrates the charge variation to obtain equation (34).

The proposed algebra is then evaluated by pulling the canonical symplectic form back to the constraint surface. Equations (35)–(40) claim that its double contraction with two residual shifts equals the bulk kernel paired with two zero modes and hence vanishes. The resulting charges are said to form an infinite-dimensional Abelian algebra without central extension. Section 4 briefly discusses matching the future and past null patches and writes the schematic flux identity (41). It also suggests that the same mechanism applies at a finite null boundary such as the BTZ horizon.

The organizing question is worthwhile. Null Hamiltonian systems do require care with zero modes, differentiability, corner data, and the distinction between proper gauge transformations and charged boundary symmetries. The paper is also concise and its tables could become a useful comparison among examples. At present, however, the formal argument bypasses exactly those points. Most examples are quoted from the authors' earlier work or from a manuscript in preparation, and the new general conclusion is not supported by a complete phase-space construction. A revised work would need to rebuild the analysis from a specified functional phase space and verify the claims in at least one example from beginning to end.

Logical structure and principal assumptions

The conclusions rely on the following chain of assumptions.

- The discrete matrix carried by $\Omega_{\alpha\beta}$ is called second class and locally invertible, while the same distributional kernel, viewed as an integral operator, is assumed to possess the modes in equations (2)–(4).
- The source J_α in equation (8) is assumed to lie in the range of this singular operator, so that a particular solution exists and the only consequence of the kernel is the arbitrary term in equation (10).
- A homogeneous multiplier ambiguity is taken to define a canonical symmetry generated by equation (11), without separately checking preservation of all constraints, the total Hamiltonian, or the action.
- Boundary conditions are assumed to make the improved generator differentiable and the charge integrable. Equations (32)–(33) fix the leading coefficients, but no complete asymptotic or horizon phase space is supplied.
- The reduced Hamiltonian vector field is assumed to have only the two components displayed in equation (35), even when f_α and $U^\alpha{}_I$ depend on other canonical fields.
- Pairing two exact bulk zero modes with Ω is assumed to compute the full improved charge bracket. This discards integrations by parts, surface pieces of the pulled-back symplectic form, and possible corner contributions.
- The charges are treated as physical observables before quotienting by the ordinary Maxwell or Yang–Mills Gauss symmetries, and their u-dependence is called a soft flux without an explicit evolution equation.

None of these steps is automatic in a constrained system with boundary. They must be established together because changing the allowed smearings or boundary conditions can change the kernel, constraint classification, differentiability, and central term at once.

Major comments

1. **The second-class classification and the residual kernel are not reconciled.**

   Equation (1) defines the Dirac matrix as a distributional integral operator. Its inverse is therefore characterized by a convolution that produces a delta distribution, after a domain and boundary conditions have been chosen. The statement following equation (1) that it is invertible "at fixed x, x'" is not an alternative notion of invertibility relevant to the Dirac algorithm. If the smearings in equation (2) are admitted and satisfy equation (3), the full operator has a kernel and no two-sided inverse on that space. Saying that the constraints remain second class in the discrete label α does not resolve this conflict.

   The authors should define the function space of fields and smearings, the boundary pairing, and the boundary conditions used to define the adjoint and kernel of Ω. They should then classify the *smeared* constraints. In particular, the combination in equation (11) has weakly vanishing brackets with the displayed constraint sector before boundary improvement; it must be explained whether this is a first-class direction, an improper gauge transformation, a degeneracy removed by boundary conditions, or a direction that pairs with additional constraints. A finite-dimensional analogy or a regulated radial example would make the distinction precise. Equations (2)–(5) cannot by themselves support both invertibility and a nonzero kernel.

2. **The multiplier equation needs a compatibility condition, and a multiplier zero mode is not yet a symmetry.**

   For a singular antisymmetric operator, equation (8) has a solution only if J_α is orthogonal to every left zero mode, with boundary terms included. Schematically, the missing condition is a projected equation $P^T J = 0$. Depending on the model, it may be an additional constraint, a boundary equation of motion, or a restriction on admissible data. The paper assumes a particular solution $\bar{\lambda}(J)$ in equation (10) without deriving this condition. The simultaneous assumption that there are no further constraints therefore removes a possible output of the Dirac–Bergmann algorithm by fiat.

   Even if equation (10) is solvable, its homogeneous part only shows that the multipliers are underdetermined. To establish a symmetry, the authors must verify the transformation of every canonical variable, preservation of all constraints and boundary conditions, and the condition on $\dot{G}_Q + \{G_Q, H_T\}$, including any required u-dependence of the parameter. If this expression is a boundary flux rather than zero, that flux should be derived. The distinction is essential because the arbitrary angular shifts in equation (24) generally change angular field strengths; their zero energy and "soft" character do not follow from the radial zero-mode equation alone.

3. **Equations (35)–(39) do not give a valid reduced-phase-space derivation.**

   There is first a definite sign error. From equation (16), $\chi_\alpha = \pi_\alpha + f_\alpha$, tangency to the constraint surface gives

   $$0 = \delta_\varepsilon \chi_\alpha = \delta_\varepsilon \pi_\alpha + \int \mathrm{d}^d x' \, \frac{\delta f_\alpha}{\delta \psi^\beta(x')} \eta^\beta(x'),$$

   so equation (36) should contain a minus sign. This sign propagates into the double contraction in equation (39).

   More importantly, equation (35) is not the full Hamiltonian vector field. The functions f_α can depend on every member of Ψ^Λ, not only on ψ^α. In Maxwell theory f_A contains A_r, and in Yang–Mills theory the zero-mode map is itself built from A_r. The generator consequently acts on momenta conjugate to those other fields, even if the fields themselves do not shift. Those components contribute to the canonical symplectic form. The authors should solve the constraints or construct the Dirac bracket explicitly, pull back the complete symplectic form including boundary terms, and only then identify the vector field satisfying equation (37). At present equation (37) is asserted, not derived from the displayed phase space.

4. **The vanishing central term is obtained by deleting the terms that can produce it.**

   Equation (12) says that the bulk pairing vanishes only "up to possible boundary terms." Regge–Teitelboim improvement is introduced precisely because those terms are nonzero. Equation (39) therefore cannot replace the full improved bracket by $\eta_1 \Omega \eta_2$ and set it to zero. After solving the constraints, integration by parts transfers radial derivatives in f_α to a surface piece of the reduced symplectic form that need not annihilate a bulk zero mode.

   The Maxwell–Pontryagin row makes the issue concrete. Equations (25), (31), and (34) imply

   $$\delta_{\varepsilon_2} Q[\varepsilon_1] = \int_{\partial \Xi} \mathrm{d}^2\varphi \, \bar{k}^r_{AB} \, \varepsilon_1^A \varepsilon_2^B, \qquad \bar{k}^r_{AB} = \frac{\epsilon\sqrt{\gamma}}{e^2}\gamma^{AB} - \frac{\theta}{e^2}\epsilon^{AB}.$$

   Its antisymmetric θ-dependent part is a nonzero candidate central bilinear. The fact that it drops out of the *bulk* operator in Table 1 is compatible with its survival at the boundary. Only the full improved bracket, including gauge identifications and corners, can determine whether it vanishes. Equation (4) cannot; equations (39)–(40) must be rederived.

5. **The proposed charges have not been shown to be observables on the physical gauge-theory phase space.**

   The paper explicitly omits the Maxwell and Yang–Mills Gauss generators before equation (24). Yet equation (34) is linear in A_A in these examples and is not manifestly invariant under $A_A \mapsto A_A + \partial_A \lambda$ or its non-Abelian analogue. A functional that changes along a proper gauge orbit is not an observable on the reduced phase space. The allowed gauge parameters, their surface charges, the transformation of ε^I, and the mixed brackets with the proposed shifts all have to be specified. In Yang–Mills theory this also requires the adjusted bracket appropriate to the field-dependent radial transporter. It may be that a restricted, gauge-invariant subset survives, or that the new shifts extend the usual large-gauge algebra; the current truncation cannot decide this. This analysis is also necessary to explain the relation to the Kac–Moody algebras invoked in Section 1, since the paper's final algebra is instead Abelian and centerless.

6. **Boundary conditions, finiteness, integrability, and dynamics must be demonstrated in the individual examples.**

   Equations (32)–(33) fix selected leading coefficients, but they do not define an asymptotic phase space. The scalar entry contains compensating powers of r, the three-dimensional Maxwell zero mode grows as $\sqrt{r}$, and the Yang–Mills transporter is field dependent away from infinity. The manuscript should state falloffs for all fields and variations, show that the symplectic form and equation (34) are finite, and verify that the boundary conditions are preserved by the residual shifts and Hamiltonian evolution. Possible symplectic flux and corner terms must be included in the integrability analysis rather than set to fixed background values without justification.

   Equation (41) is the identity obtained by defining $\mathcal{F} = \dot{Q}$; it is not yet a flux-balance law derived from the equations of motion. At least one scalar example and one gauge example should include the explicit charge, its u derivative, matching conditions, flux, and bracket. That would also make precise whether these are conserved soft charges, quasilocal edge coordinates, or nonconserved boundary observables.

7. **The finite-horizon claim and the journal-level novelty are not established.**

   The only finite-distance discussion is the qualitative BTZ paragraph at the end of Section 3. No gravitational canonical variables, horizon constraints, zero-mode map, boundary conditions, charge, or algebra are computed. The endpoint statement in Section 4 does not substitute for a horizon phase-space analysis. The abstract and conclusion should either be restricted to the demonstrated Minkowski matter examples or include a complete finite-horizon calculation.

   More broadly, Section 3 presents the examples as a summary of earlier work, including an unpublished manuscript, while the general formalism is attributed to the authors' prior papers. For PRD or JHEP, the authors need to isolate a new theorem, new charge algebra, or new physical consequence and compare it directly with the established null-surface canonical, asymptotic-symmetry, edge-mode, and horizon-symmetry literature cited in the paper. A corrected Maxwell–Pontryagin central-term calculation or a full BTZ example could provide such content. Without it, the manuscript reads as a useful conference overview rather than a substantiated journal article.

Minor comments

1. The inverse mentioned after equation (1) should be written with its integration variables and delta distribution. This would expose the role of the operator domain and avoid the misleading phrase "at fixed x, x'".

2. Define the measure, transpose, and boundary pairing associated with $P^{\alpha}{}_I$ in equations (2)–(5). For a field-dependent map, also state its rank and whether the index I labels a complete kernel basis.

3. The phrase "global zero mode" is potentially confusing because the parameters are arbitrary functions of the angular coordinates. Explain that "global" means constant only along the radial direction, if that is the intended usage.

4. The claims in Section 1 that these modes do not change the energy and are Goldstone-like should either be proved from the Hamiltonian and boundary conditions or presented as motivation rather than established properties.

5. Equations (26)–(28) alternate between $U^a{}_b$ as an adjoint transporter and U as a group-valued Wilson line. Introduce a group element $g(r, \varphi)$ and write $U = \mathrm{Ad}_g$, with the path ordering and sign convention checked explicitly.

6. Section 3 lists scalar–Maxwell and axion–Maxwell among the examples, but neither appears in Tables 1 or 2. Include their data or remove them from the list.

7. Define the star bracket in equation (38): is it a Dirac bracket, the Poisson bracket induced by the pulled-back symplectic form, or a quotient bracket after gauge reduction? These need not coincide before all degeneracies and boundary modes are treated.

8. The notation $\bar{U}$ in Table 2 should distinguish a finite limit from a fixed leading radial coefficient. In the scalar and three-dimensional Maxwell rows the displayed factors themselves scale with r.

9. A manuscript in preparation cannot carry the evidentiary burden for the Maxwell null-boundary example. The essential derivation should be included here or replaced by a public reference.

10. The tables are useful, but the index placement of $k^r_{\alpha\beta}$, σ^{AB}, and $U^\alpha{}_I$ should be made consistent. This matters for identifying symmetric versus antisymmetric contributions to the boundary bilinear.

11. The prose is generally readable, but the repeated description of the shifts as both "asymptotic," "global," and "quasilocal" obscures three different notions. Each term should be tied to a precise domain, boundary, and matching prescription.

Publication recommendation

The manuscript identifies a real conceptual issue and could become a useful contribution if the null zero-mode sector is treated within a complete constrained phase space. The present proof of the Abelian centerless algebra is not correct as written, and the physical status of the charges is not yet established. I therefore do not recommend publication in PRD or JHEP in the current form. A substantially new version that resolves the operator-domain and Dirac-consistency questions, recomputes the reduced bracket including boundary terms and ordinary gauge symmetries, and supplies a fully worked example could merit fresh consideration.